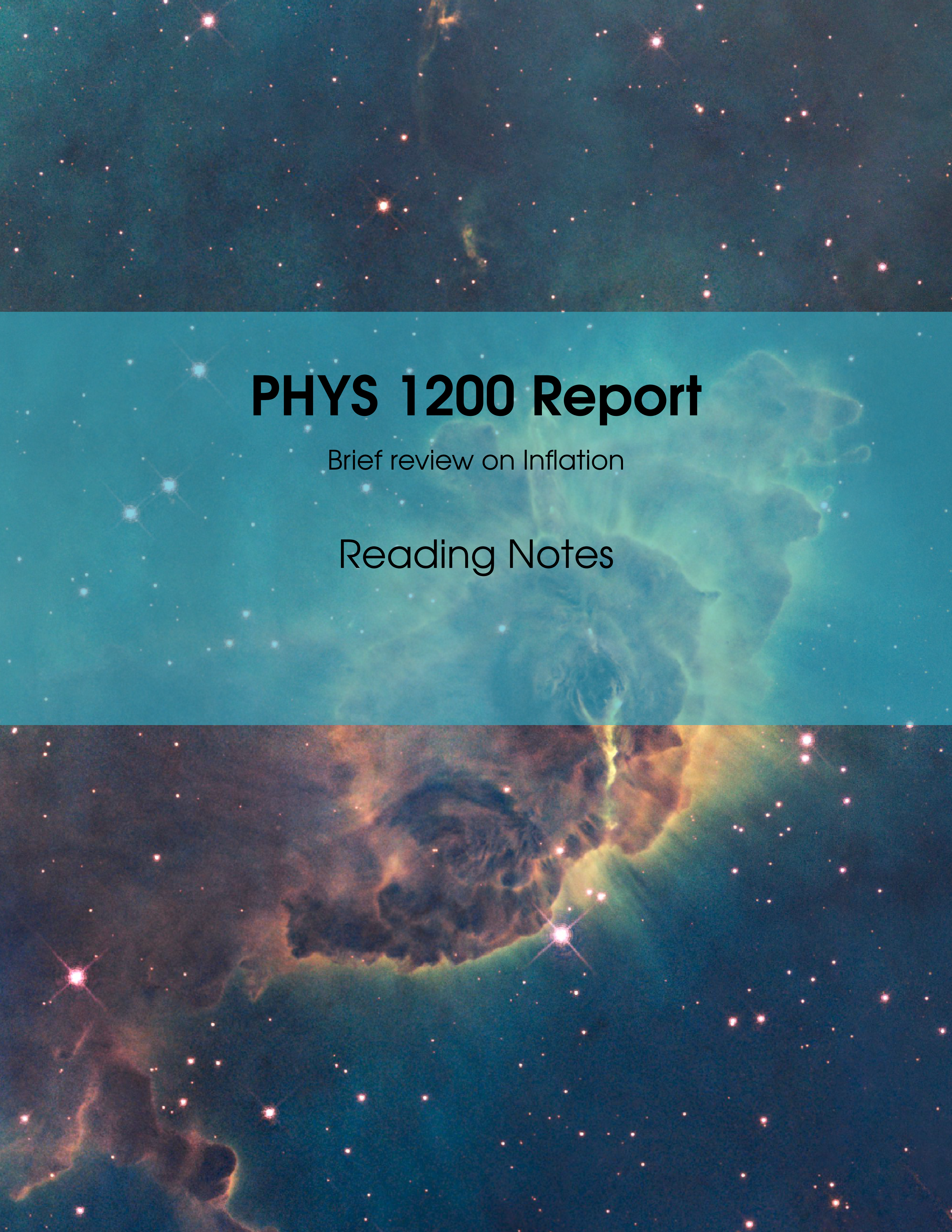

PHYS 1200 Report

Brief review on Inflation

Reading Notes

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1. A Glimp at Inflation

1.1 Universe is expanding

I can't imagine what the universe would be like if it didn't expand. Olbers' paradox tells us that if the universe were static, night time would be as hot as the day time. At the same time, Hubble's observations also tell us that the universe is expanding.

The fact that theIt's difficult to imagine what the universe would be like if it didn't expand. Olbers' paradox tells us that if the universe were static, the night sky would be as bright as the day, due to an infinite number of stars in every direction. However, observations made by Hubble and other scientists have confirmed that the universe is indeed expanding.

The expansion of the universe is now an accepted fact, much like the acceptance of the heliocentric model in the 18th century was a natural progression of understanding.

1.2 Why universe is expanding?

To understand why universe is expanding, let's first consider what we can observe in the universe on a cosmological scale. The universe appears to be isotropic (the same in all directions) and homogeneous (uniform) on large scales. From this observation, we can deduce the following fundamental principles of cosmology:

> **Theorem 1.2.1 — Fundamental principles of cosmology.** The spatial distribution of matter in the universe is equally distributed and isotropic when viewed on a large enough scale.

However, this property only holds for a specific class of observers, and these observers can be described using the FLRW metric to represent the structure of spacetime. Through this specific spacetime structure, we can derive an expanding universe.

1.2.1 What is the special obeserver?

If we consider an observer, the observational properties mentioned above may not hold. One can imagine that such an observer would see approaching stellars and planets, leading to a loss of

Homogeneity and Isotropy in their reference frame. Therefore, the basic principles of cosmology only hold for a specific class of observers.

According to Baumann, this specific class of observers, known as comoving observers, observe **orthogonal** slicings (spacelike hypersurfaces of constant cosmic time) and threading (worldlines of fixed spatial coordinates) which differs space and time obviously. These observers perceive a 0-momentum density. And they are free-falling observers, and as a result, thus they perceive a world that is Homogeneous and Isotropic. Therefore, this class of observers needs to be distinguished from other observers.[Bau09]

More than 90% cosmology would focuses on this class of observers, remember that! These observers perceive the universe to have several simple properties. The general form of the energy-momentum tensor is:

$$T^{\mu}_{\nu} = (\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - \bar{P}\delta^{\mu}_{\nu}$$

However, for this class of observers, the energy-momentum tensor is diagonalized.

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}$$

For further details on the energy-momentum tensor, please refer to Appendix: Energy-Momentum Tensor.

1.2.2 Why the observer prefer to use FLRW metric?

As mentioned above, in the perspective of such special observers, the universe possesses Homogeneity and Isotropy. While using the variables x, y, and z may not reflect this symmetry well, using spherical coordinates can effectively utilize this symmetry.

Hence, introducing the FLRW metric, where the parameter k corresponds to the gaussian curvature term of the universe:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$$

$$ds^2 = a^2(\tau)\left[d\tau^2 - (\chi^2 + S_k^2(\chi)d\Omega^2)\right]$$

Note:

$*$1. Since we consider an Isotropic universe, the $d\Omega$ term can be omitted.

$*$2. One advantage of using spherical coordinates is that it introduces the radial coordinate r into the metric, resulting in non-trivial terms when differentiating $\partial g_{\mu\nu}$. This means that the connection $\Gamma^{\rho}\mu\nu$ is no longer zero, and the Ricci scaler $\mathscr{R}$ is non-trivial.

$*$3. Despite appearing complicated, these coordinates are actually very useful because we can quickly see that the universe's expansion can not be the simple case with constant velocity. (Proof of this claim is coming soon)

1.2.3 Why is the universe described above expanding?

In the context of cosmology, the expansion of the universe is described by the friedmann equations, which are derived from the Einstein field equations. The most fundamental equation to describe the universe is the Einstein field equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

By considering a Homogeneous and Isotropic metric and calculating the connections and Riemann curvature tensor, we can derive the friedmann equations. Forget things like Einstein equation, we can play with Cosmology simply using the friedmann equations. (Detailed derivation can be found in baumann's lecture note)

friedmann Equation

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \end{aligned}$$

Here, $a(t)$ represents the scale factor of the universe, and ρ and P are the density and pressure, respectively, in the energy-momentum tensor.

A useful corollary of above equations is:

$$\dot{\rho} + 3H(\rho + P) = 0$$

Definition 1.2.1 — barotropic fluid. A kind of fluid whose density depends only on the pressure we can write $\rho = P/w$

Considering that the cosmic fluid is a barotropic fluid.This allows us to express the density as $\rho \propto a^{-3(1+w)}$. Substituting this relation back into the friedmann equations, we obtain the scale factor as a function of time for different equations of state:

$$w_{\text{matter}} = 0, \quad w_{\text{radiation}} = \frac{1}{3}, \quad w_\Lambda = -1$$

$$a(t) \propto \begin{cases} t^{\frac{2}{3}} \\ t^{\frac{1}{2}} \\ e^{Ht} \end{cases}$$

In the conformal form, the scale factor can be expressed as:

$$a(\tau) \propto \begin{cases} \tau^2 \\ \tau \\ \tau^{-1} \end{cases}$$

This is spatial scale expanding eqn. **Note:**

$*$1. The expansion of the universe is a consequence of solving the Einstein field equations under certain conditions, and it leads to the emergence of the scale factor $a(t)$ as a variable. Discussing whether stars or even humans expand with the universe is not meaningful without first solving the

Einstein field equations. Once the symmetry is lost, it becomes difficult to separate such a variable. Whether the same results can be obtained using Cartesian coordinates requires further calculations.

$*2$. The equation of state should be emphasized for its importance. By studying the friedmann equations, we can understand that the expansion of the universe is driven by different types of material, each contributing differently to the expansion.While universe dominated by different kinds of material, an important effect is the changing causal structure of spacetime. This will be further explored in the context of inflation.

1.3 How does the universe expand exactly?

To begin with, let's understand how to describe the geodesic/or say world-line of cosmic scales, then, we will address two difficulties in proving classical cosmic expansion. Finally, we will introduce the concept of inflation.

1.3.1 The $\chi - \tau$ Diagram and Horizon

It is advisable to use the FLRW metric to describe the universe, as this metric retains the symmetries of the cosmological principles, making it more convenient to work with. Since we consider the universe to be isotropic, we can neglect the angular terms. Therefore, the metric simplifies to $\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \mathrm{d}\chi^2\right]$.

Cosmologists find this coordinate system very convenient for plotting! Due to the correspondence of the null geodesic ($\mathrm{d}^2 s \equiv 0$) with a 45° straight line, it becomes extremely easy to draw the light cones.(Just think about it, if this line is sinuous, how ugly it would be)

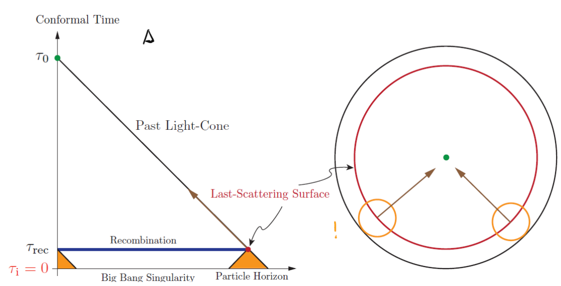


Figure 1.1: refer from [Bau09]

1.3.2 Two Problems within the classical universe expansion

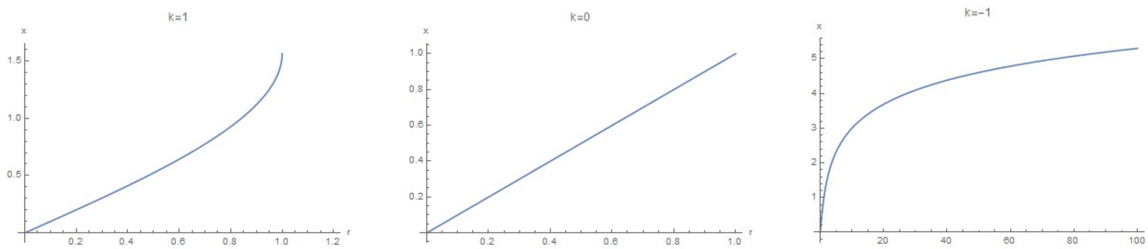
1. The Horizon Problem

In this 2D $\chi - \tau$ diagram, the light cones form an 45 degree right triangle, and the causal domain corresponds to the area covered by the small orange triangles. It is visually clear that the two orange triangles do not overlap. However, what does this imply in physics? It means that before this time, there was no causal connection between these two points.

Sounds a little bit terrifying isn't it, if I tell you that the recombination epoch corresponds to the blue line in the figure, it means that in the early universe, at large scales, there was no causal contact between different regions. Therefore, it seems impossible for thermal equilibrium and heat exchange to be established. However, our observations indicate that the temperature distribution of the cosmic microwave background (CMB) is remarkably uniform. This poses a puzzling challenge known as the Horizon problem.

Does the overlap of comoving coordinates represent an intersection of physical coordinates?Indeed, it does.

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}$$



(a) $k=1$ for closed universe (b) $k=0$ for flat universe (c) $k=-1$ for open universe

Figure 1.2: We can tell that $\chi \sim r$ has 1-1 correspondence in all situation

2.Flatness Problem

The evolution of the universe can be primarily determined by two friedmann equations, and solving this system of differential equations requires knowledge of the initial conditions. Under the assumption of a classical inflationary universe, how can we determine the initial conditions?

The observed large-scale uniformity of the universe suggests that the spatial distribution of density was also uniform in the early universe.Since a spatially uniform density distribution is unstable, density perturbations generated in the early universe are stretched during cosmic expansion and eventually form the localized structures observed today. Therefore, this assumption has some validity. [Bau09]

The initial velocity distribution of the early universe needs to be finely tuned. If the velocity is too large, the current universe would become more sparse; if it is too small, the universe would prematurely enter the recollapse phase. In addition, the difference between kinetic and potential energy is related to ρ and P, and the energy-momentum tensor affects the local curvature of the universe.

Therefore, to obtain a universe with $k \approx 0$ as observed today, the initial velocity distribution needs to be finely tuned.However, physicists are generally not fond of finely tuned models, and this issue is known as the flatness problem.

To be more specific, let's see some number to have feeling of the meaning of fine-tuning. Since the universe is pretty flat and can even be approximate to a Euclidean space, $|\Omega_k|$is pretty small. $|\Omega_k| \propto \rho a^2$, since ρ is rapidly decreasing in classic universe expanding theory, thus in early universe it shall be extremely small to correspond to now-a-days' observation.

$$\begin{aligned} |\Omega(a_{\mathrm{BBN}})-1| &\leq O(10^{-16}) \\ |\Omega(a_{\mathrm{GUT}})-1| &\leq O(10^{-55}) \\ |\Omega(a_{\mathrm{pl}})-1| &\leq O(10^{-61}) \end{aligned}$$

Another way to understand that since ρ is approximate constant during inflation, thus $|\Omega_k|$ can be heavily suppressed by a during inflation and give us more freedom to adjust the initial value of $|\Omega_k|$

Meanwhile a more mathematical proof of Flatness problem is as follow:

$$\frac{d\Omega(a)}{d\ln a} = (1+3w)\Omega(a)(\Omega(a)-1)$$

It is obviously that $\Omega = 1$ is an unstable fixed point under S.E.C

Definition 1.3.1 — SEC. strong energy condition means $1+3w>0$

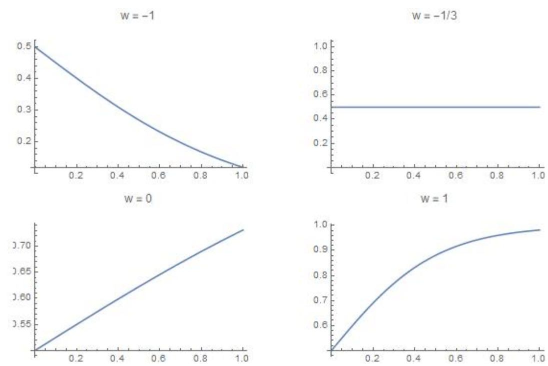


Figure 1.3: Attractive solution

1.3.3 Inflation as Solution

One method to address the issues mentioned above is called the inflationary model. The inflationary model aims to solve the Horizon problem by allowing for a larger amount of Conformal time, which provides the universe with more history in terms of conformal time. This allows the past light cones of every point on the observed cosmic microwave background (CMB) within the current observable universe to have intersected in the sufficiently distant past.In terms of solving the Flatness problem, the inflationary model achieves it by causing the Hubble radius to shrink.

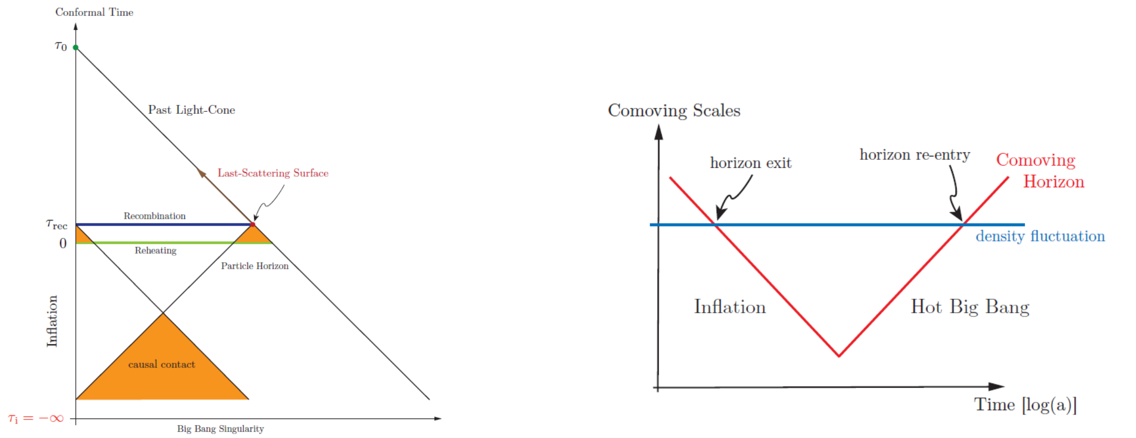
1.3.4 Are the two solutions mentioned above consistent?

In another word, can we use the diagram that solves the Flatness problem to explain the Horizon problem?

The answer is yes. Rotate the 2D diagram around the time axis to create a three-dimensional (2+1) image. This brings us closer to our (3+1)-dimensional spacetime and provides a better visual understanding.The orange region represents the area in the cosmic microwave background (CMB) that satisfies the cosmological principle **at present**

The red circles represent the "real-time" causally connected regions defined by the comoving observer at different times.(The red circle marked "now" within the orange region represents the region where real-time causal connections are happening at present, while the orange region surrounding the red circles represents the observed CMB that cannot have "real-time" connections.)

This diagram tells us that we observe the CMB, which used to have "real-time" causality within a certain time period since it reached thermodynamic equilibrium. Although it gradually lost the



(a) More history (b) Solving flatness problem

Figure 1.4: Shrinking Hubble radius

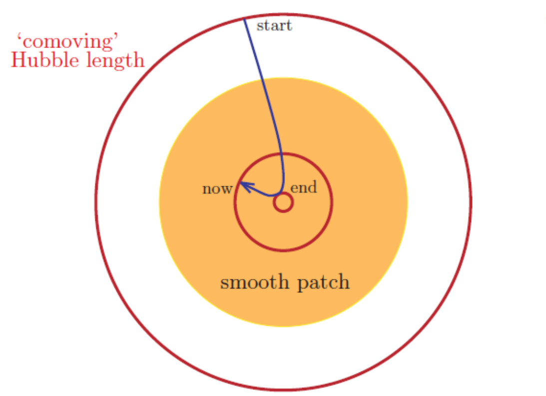


Figure 1.5: refer from [Bau09]

"real-time" causal connections later on as universe expanding, the currently observed universe still satisfies homogeneity and Isotropy.

•How to understand the Radius and Horizon in this diagram?

With those diagram in mind, let's discuss 2 different Horizon. To fully understand this diagram, we need to grasp the concept of the Horizon and the "real-time" boundary. What is the observed boundary?Let's start with the answer: the first corresponds to the Hubble radius, and the second corresponds to the Particle Horizon.

Definition 1.3.2 — Particle Horizon. Particle horizon, also called the cosmological horizon, the **comoving horizon**, or the cosmic light horizon.

Definition 1.3.3 — Hubble Radius. Hubble volume or Hubble sphere, Hubble bubble, subluminal sphere, causal sphere, and sphere of causality.

Now that we understand their semantic definitions,before that let's gain a feeling about these two concepts through some facts. I think it would be helpful in understanding the discussion below.

The universe is expanding.
The universe is currently undergoing accelerated expansion.
The Hubble parameter is decreasing.

1.3.5 Detailed explaination on two kinds of horizon

1. Hubble radius

∗Can we see the entire universe that is infinitely far away?

No, because the universe is expanding. Neglecting peculiar motion, according to the Hubble law, we know that $v = r \cdot H$. Therefore, the sphere defined by $r = \frac{c}{H}$ represents the positions that are receding from us at the speed of light.

∗What kind of emitted light in the whole life of the universe can enter our eyes?

(If we really want to delve into it, considering only the light emitted at the exact boundary of this sphere, there are some issues related to the simultaneity in special relativity. However, we can avoid this problem.)

If we consider outside the sphere, the recession velocity is definitely greater than the speed of light, so it cannot enter at that time. If we consider inside the sphere at $r - \varepsilon$, where the velocity of photons is slightly greater than the recession velocity, they are allowed to propagate inward. Therefore, what we can see is the light emitted at the comoving coordinate $r = \frac{1}{aH}$. **This distance is the Hubble radius**.

We can consider the process of photons entering from near the Hubble sphere relatively fast—during this process, $a(t)$ changes only a little. In other words, the background of the entire universe does not change significantly, at least not by several orders of magnitude. Therefore, we can regard the light transmitted from near the Hubble sphere as "instantaneous." This means that the timescale of the physical processes we study belongs to a smaller order of magnitude compared to the scale of background evolution.

∗Why is it said that $a(t)$ changes little during this process?

We define the extension time as $t_H \equiv H^{-1} = \frac{dt}{d\ln a}$. Therefore, another meaning of the Hubble radius is the distance that photons can propagate during an extension time. $c \cdot t_H = 1 \cdot \frac{1}{H}$. In the case of a Λ-dominated universe, $a \sim e^{Ht}$, so an extension time corresponds to an approximate doubling of the scale factor.

2. Particle Horizon

Now let's consider another type of timescale—the timescale of the Particle Horizon, which represents whether photons can reach a certain place "in their lifetime." This time is very long and corresponds to the consideration of the scale factor $a(t)$ changing over time. We need to take into account not only the expansion of the universe but also its accelerated expansion, where $\dot{H} \leq 0$.

So, when we switch to different moments and look at the Hubble Radius, the visible range is different, which is given by $r = \frac{1}{H}$. Therefore, in the physical reference frame, the radius appears to be increasing. This means that photons emitted in early time sup-Hubble scale can be engulfed by the expanding Hubble Sphere, allowing us to see light from more distant stars(Outside hubble radius).

∗How can we describe this Horizon?

We can understand this Horizon as the distance traveled by photons throughout their lifetime, coincidentally, in conformal coordinates, $\Delta\chi = \Delta\tau$.

$$\chi_{\rm ph}(a) = \frac{2H_0^{-1}}{(1+3w)}\left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}\right] \equiv \tau - \tau_i$$

3. Summary:

The Hubble radius corresponds to the causal structure at a specific moment and describes an instantaneous property. The Particle Horizon describes the lifetime of a particle. Therefore, it is apparent that $\chi_{PH} \geq r_{Hubble}$.

What does this 2D plot tell us?

The solution in solving two cosmological problems all at once is simply an operation that bending curves(straight line) in the $\chi - \tau$ diagram above. It may seem unreasonably simplified, but this model is consistent with current observations which will be discussed in detail in Part 4. Now accepting that this model is solid, let's go and explore what more this model could tell.

1.4 What is Inflation?

The initial definition of inflation can be thought as an operation in the 2-D diagram mentioned above, we can have more concrete interpretation derive from it. Which means Inflation have several equivalent definitions:

•Decreasing Hubble sphere:

This can be directly obtained from the image by Baumann.

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0$$

•Accelerated expansion:

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = \frac{-\ddot{a}}{(aH)^2}$$

•Inflation parameter $\varepsilon < 1$: It can be derived from the accelerated expansion of the universe.

$$\frac{\ddot{a}}{a} = H^2\left(1 + \frac{\dot{H}}{H^2}\right) > 0$$

Definition 1.4.1 — Inflation parameter ε.

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \equiv -\frac{d\ln H}{dN}$$

Here, $-\frac{\dot{H}}{H^2}$ is a commonly encountered form, so we call it ε, which represents the change in the Hubble parameter corresponding to a doubling of the volume (increasing by one e-fold) during cosmic expansion. Here, N is called the e-folds, representing the logarithmic transformation of the scale change: $dN = Hdt = d\ln a$.

Therefore, Inflation is equivalent to:

$$\varepsilon < 1$$

•Negative pressure:

From the friedmann equation, we know that $\ddot{a} > 0 \Rightarrow$

$$P < -\frac{1}{3}\rho$$

The most intuitive definition that can be obtained from the curved lines on the $\chi - \tau$ plot is that there is a segment with a negative slope. In this plot, the horizontal axis represents $ln(a)$, and the vertical axis represents $ln((aH)^{-1})$. The slope is given by $\frac{d\,ln((aH)^{-1})}{d\,ln(a)} = -\frac{\ddot{a}a}{\dot{a}^2}$, and by substituting into the friedmann equation, we obtain $\frac{\rho+3P}{2\rho} = \frac{1+3w}{2}$. Therefore, the conditions for Inflation correspond to $w \leq -\frac{1}{3}$, making this definition consistent.

•Slowly-varying Hubble parameter:

$\varepsilon = -\frac{\dot{H}}{H^2}$ indicates that the Hubble parameter is slowly varying.

•Quasi-de Sitter expansion:

Definition 1.4.2 — de-Sitter space-time. de Sitter space-time corresponds to perfect inflation: $\varepsilon = -\frac{\dot{H}}{H^2} = 0 \rightarrow a(t) = e^{Ht}$

Therefore, the metric becomes: $\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2Ht}\mathrm{d}x^2$ Since this spacetime describes eternal inflation,which is no-physical, we can only use it as an approximation when $\varepsilon << 1$.

•Constant density:

$\left|\frac{d\ln\rho}{d\ln a}\right| = 2\varepsilon$, so it can be approximately assumed that

$$\rho \approx const$$

Summary

$$\frac{d}{dt}(\frac{1}{aH}) < 0$$

$$\ddot{a} > 0$$

$$\varepsilon < 1$$

$$P < -\frac{1}{3}\rho$$

$$\dot{\varepsilon} \ll 1$$

$$a = e^{Ht}\text{quasi de-Sitter}$$

$$\rho \approx const$$

Inflation theory addresses two cosmological problems: the Flatness problem and the Horizon problem. A clever bent on Hubble radius curve on the $\chi - \tau$ plot solve both problems at once. Based on this solution, we derived a series of equivalent definitions for Inflation. But can this operation really be achieved? Or say what are the feasible corresponding operations in reality?

The friedmann equation tells us that the expansion of the universe is based on matter, and different types of matter yield different results– different causal structures in space-time. Therefore, exploring the physical nature of inflation is equivalent to asking whether there is corresponding matter that allows for inflation in the universe. Fortunately, we can find such material.

1.4.1 What is the physical requirements for Inflation?

Here, we have to mention the need for the barotropic fluid assumption.

Theorem 1.4.1 — barotropic fluid assumption. In cosmology, it is believed that the universe is filled with a barotropic fluid. Therefore, during certain time periods, such as the radiation-dominated era, we obtain a constant equation of state, which means $\frac{P}{\rho} \equiv w$ during this stage.

Recalling that P and ρ are obtained from the diagonal terms of the energy-momentum tensor, solving the inflation problem becomes finding the appropriate substance that satisfies the required energy-momentum tensor.

In classical physics, the equation of state is typically characterized by three values: $w = 0$, $w = \frac{1}{3}$, and $w = -1$.We classify the "stuff" in the universe as non-relativistic matter, relativistic matter (radiation), and the cosmological constant based on their equation of state. However, none of these values satisfy our requirements. We need a new energy-momentum tensor, which implies that we need a new action.

Quantum field theory provides various novel actions. Let's consider the simplest case : real scalar field. Surprisingly, we find that this field has enough degrees of freedom to allow us to achieve the desired w:

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right)$$

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right)$$

Note:

The ϕ we describe is the ϕ field observed from the comoving observer's perspective,it can thus possesses isotropic properties. Therefore, when calculating $T^\mu{}_\nu$ for the comoving observer, we can neglect $\partial_i \phi$, and we obtain:

$$T^0{}_0 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$T^i{}_i = P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$w = 1 - \frac{2}{1+\frac{\dot{\phi}^2}{2V}}$, where we denote $\frac{\dot{\phi}^2}{2V}$ as r.

When $V = 0$, we can obtain $w = 1$. Therefore, we can see that the action of the real scalar field already has enough flexibility to obtain any desired w. Complex field or spinor field has even more DOF but I guess that would be just too much. Meanwhile we have already obtained a promising Action, let's just focus on this real sclaer field and further analyze it.

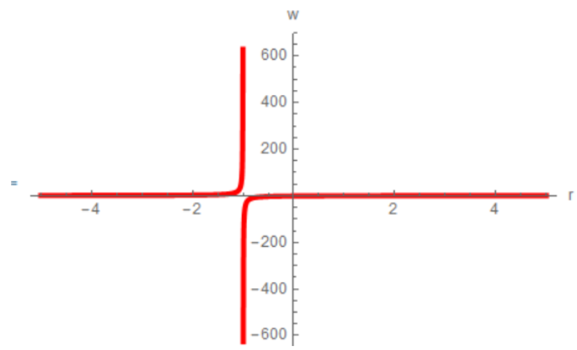


Figure 1.6: w can be achieved through scaler field

1.5 What dynamic does this sclaer field has?

In principle, to describe a complete system, we need to obtain a complete action. However, this would involve the coupling between gravity and scalar fields. So, as a preliminary approach, we can construct the system's action without considering Ricci scaler $\mathscr{R}$. This approach allow us to qualitatively obtaining the dynamics of the system and give constraints on $V(\phi) aswell$.

1.5.1 Obtaining the equation of motion for ϕ

We obtain the energy-momentum tensor from the action, which gives us P , ρ. Substituting these quantities into the friedmann equation allows us to solve the evolution equation in a matter-dominated universe. Therefore, we have

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right]$$

Definition 1.5.1 — Planck mass M_{pl}. $M_{\rm pl}$ corresponds to the mass when a particle's de Broglie wavelength is equal to its Schwarzschild radius, and it reflects an upper limit on particle mass. $\sqrt{\frac{c}{8\pi G}} = 2.4 \times 10^{18}, \mathrm{GeV}$

Equation (1) can be substituted into the second friedmann equation:

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm pl}^2}$$

Taking the derivative of equation H^2:

$$2H\dot{H} = \frac{1}{3M_{\rm pl}^2}\left[\dot{\phi}\dot{\phi} + V'\dot{\phi}\right]$$

Combining these equations, we obtain **Klein-Gordon eqn**

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

1.5.2 Constraints on $V(\phi)$

Let's talk about Inflation's condition:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\text{pl}}^2 H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{\frac{\dot{\phi}^2}{6}+\frac{V}{3}} < 1 \Rightarrow \frac{\dot{\phi}^2}{2} < V(\phi)$$

Under this approximation, we have

$$H^2 \approx \frac{V}{3M_{\text{pl}}^2}$$

ince observations from the CMB tell us that inflation must have occurred for at least 60 e-folds, so

$$N_{\text{tot}} \equiv \int_{a_{\text{initial}}}^{a_{\text{final}}} d\,lna = \int_{t_I}^{t_E} H(t)\,\mathrm{d}t > 60$$

This tells us that the universe must undergo inflation for a sufficiently long time, meaning that ϕ cannot evolve too fast.Thus to satisfy enough inflation, we need slow-roll condition. If $\delta \ll 1$, then the above condition is satisfied.

Definition 1.5.2 — acceleration parameter. we define the acceleration parameter for the evolution of ϕ: $\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}$.

1.5.3 Inflation condition equivalence

Using inflation condition $\dot{\varepsilon} ll 1$, we have η which is greatly helpful in the discussion in Part 5.

Definition 1.5.3 — η.

$$\frac{\dot{\varepsilon}}{H\varepsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\varepsilon - \delta) \equiv \eta$$

We consider the conditions $\{\varepsilon, |\delta|\} \ll 1$ as singl field slow-roll inflation requires. This is a safe range where $\varepsilon \ll 1$ ensures that inflation can occur and $\delta \ll 1$ guarantees that inflation does not end too soon. Therefore, the slow-roll condition creates a purely inflationary environment.

Note:

$*$Under the conditions specified above, since we have "$\ll$", we have: $\{\varepsilon, |\delta|\} \ll 1 \Leftrightarrow \{\varepsilon, \eta\} \ll 1$

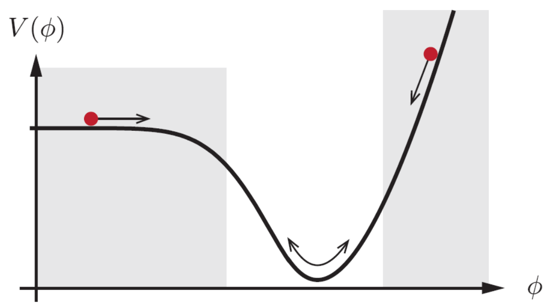
$*$The qualitative description of the slow-roll condition is shown in the following figures:

Interpreting the graph of Inflation:

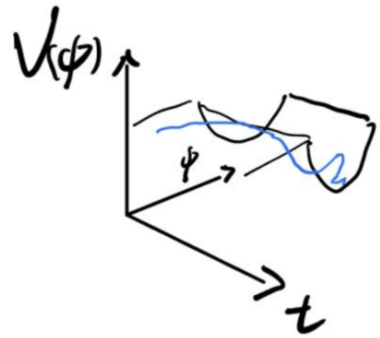
Note that we don't know what $V(\phi)$ really looks like up to this point. We can only give some properties that a Potential that satisfies Inflation should have via the slow-roll condition.

It can be seen that $\phi \approx const$ for a long time in 3D plot, this corresponds to $\frac{\dot{\phi}^2}{2} \ll V(\phi)$ during evolving. When ϕ slides down $V(\phi)$ it means the end of Inflation. The ϕ gains a lot of kinetic energy meanwhile the potential energy decreases, which broke the inflation condition and cut the Inflation.

Those figure reveals that there is a positive correlation between $\phi \sim t$ due to the slow roll condition.To be more specific the slow roll condition tells us that $\delta \ll 1 \Rightarrow \ddot{\phi} \ll \dot{\phi}$, so $3H\dot{\phi} \approx -V'$. That is to say that $\dot{\phi}$ at a given moment can be equivalently described by a position in $V(\phi)$. There is



(a) Intuitive inflation's potential 2-D



(b) Intuitive inflation's potential 3-D

another motivation in doing so. Our goal is to find a $V(\phi)$ that is satisfied by observed Inflation, and a $V(\phi)$ that generates Inflation must at least satisfy

$$\int_{t_I}^{t_E} H(t)\,\mathrm{d}t > 60$$

but the classic method suggest that we need to solve $\dot{\phi}$ entirely before we can get ε in order to compute whether or not the $V(\phi)$ satisfies, which is very cumbersome.

$$\int_{t_I}^{t_E} H\mathrm{d}t = \int_{t_I}^{t_E} \frac{H}{\dot{\phi}}\mathrm{d}\phi = \int_{t_I}^{t_E} \frac{1}{\sqrt{2\varepsilon}}\frac{\mathrm{d}|\phi|}{M_{\mathrm{pl}}}$$

But here we see that $\dot{\phi}$ has an equivalent description to $V(\phi)$.So we now establish a direct restriction of ε to V, which means that we define another set of Inflation parameter: the evolution of ϕ through the derivative of potential will save us a lot of effort.

$$\dot{\phi} = -\frac{V'}{3H}$$

$$\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\mathrm{pl}}^2 H^2} \approx \frac{M_{\mathrm{pl}}^2}{2}\left(\frac{V'}{V}\right)^2$$

$$3H\dot{\phi} + 3H\dot{\phi} = -V''\dot{\phi}$$

$$\delta + \varepsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx M_{\mathrm{pl}}^2 \frac{V''}{V}$$

Summary: Considering the basic conditions that Inflation needs to satisfy and to have enough Inflation to satisfy the observation, and then considering an approximation that is appropriate to compute Inflation, we give the restriction on $V(\phi)$

Definition 1.5.4 — Second pair parameter.

$$\varepsilon \approx \varepsilon_{\mathrm{v}} \equiv \frac{M_{\mathrm{pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1$$

$$\eta + \varepsilon \approx |\eta_{\mathrm{v}}| \equiv M_{\mathrm{pl}}^2 \frac{|V''|}{V} \ll 1$$

Note: The relations we use here are all "$\ll$"$\gg$", why we choosing an expression vague like that? This is because the whole scenario we are considering is not complete. Tink about it, the above action quantities are only describing the action quantities of ϕ, but there are also gravitational fields in the system, and both of them interact with each other.Consider minimally coupled gravity what we should have is:

$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \left[R + (\partial \phi)^2 - 2V(\phi) \right]$$

This action is much more complicated than the previous scalar field action, but gives us the full energy tensor. And the "$\gg$" or "$\ll$" vague condition allow us to use the same slow roll condition even after we change the action.

1.5.4 What can $V(\phi)$ look like?

Taking $m^2\phi^2$ as an example:

$$\varepsilon_{\mathrm{v}}(\phi) = 2 \left(\frac{M_{\mathrm{pl}}}{\phi} \right)^2 \ll 1 \Rightarrow \phi \gg \sqrt{2} M_{\mathrm{pl}}$$

•Small field inflation:
$*$Higgs-like potential:

$$V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^2 \right]^2$$

This can be generalized as $V(\phi) = V_0 \left[1 - \left(\frac{\phi}{\mu} \right)^2 \right]^2 + \cdots$ Only the small field inflation can be approximated by an expansion.

$*$Coleman-Weinberg potential:

$$V(\phi) = V_0 \left[\left(\frac{\phi}{\mu} \right)^4 \left(\ln \left(\frac{\phi}{\mu} \right) - \frac{1}{4} \right) + \frac{1}{4} \right]$$

•Large field inflation
It starts from a large value of ϕ and evolves until it reaches $\phi = 0$.
$*$Chaotic inflation:

$$V(\phi) = \lambda_p \phi^p$$

$*$Natural Inflation field:
The parameter f controls the size of $\Delta\phi$ and determines whether it is a large field inflation or a small field inflation. This field induce Axions which will be further discussed in Part 5

$$V(\phi) = V_0 \left[\cos \left(\frac{\phi}{f} \right) + 1 \right]$$

•More possibilities for inflation fields:

1. We have only considered minimally coupled gravity. It is possible to consider higher-order couplings between gravity and the inflation field.

2. Non-canonical kinetic terms can be introduced by including higher derivative terms in the Lagrangian.

3. Multiple fields can interact and contribute to inflation.

2. Quantize Inflaton Field

2.1 Quantum Properties of Scalar Field

Up to this point, we have found a substance called Real Scalar Field that has the potential to describe Inflation. We have also identified the constraints on this Scalar Field, known as the slow-roll conditions, which impose limitations on $V(\phi)$.

Since ϕ is an unobservable physical quantity, this model ultimately represents a "fabricated" field created by people. Actually that is what phenomenologist did. The picture is quite nice but it needs to be connected to and tested by experiments. So, how do we establish a connection between this field and observable physical quantities? To reveal the answer, the quantum effects of this field will predict the large-scale structure of the universe. Thus, we can further constrain the microscopic quantum field through extremely macroscopic observations.

The uncertainty principle holds at the microscopic scale, resulting in vacuum fluctuations in the density field. This leads to local density variations, causing the evolution to be locally different in time. Eventually, this results in tiny temperature fluctuations in the CMB and some local inhomogeneities in the universe.

Since we are discussing a scalar field, the observable density perturbations actually reflect the perturbations of ϕ: $\phi(\tau,x) = \overline{\phi}(\tau,x) + \frac{f(\tau,x)}{a(\tau)}$. To study the behavior of perturbations in this system, we need to obtain the action for the perturbation term.

Action for Perturbation

The evolution of the perturbation is determined by the background field $\overline{\phi}$ and $\tilde{g}_{\mu\nu}$, which is a part of the complete action. To obtain this equation, we need to expand the action.

Let's start with the action for the complete system, considering a simplified case where we only consider the action of the scalar field without coupling to gravity. The complete action should include the term Ricci scaler $\mathscr{R}$ in the action:

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[(g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - 2V(\phi)\right]$$

Under certain conditions, it can be written as

$$S=\int \mathrm{d}\tau \mathrm{d}^3x \left[\frac{1}{2}a^2\left((\phi')^2-(\nabla\phi)^2\right)-a^4V(\phi)\right]$$

Specific conditions can be found in Part 3 we'll just take this as a claim at present and think about it later: Consider the spatially flat gauge, where the spatial part is $\nabla\phi$. Note that due to the slow-roll condition, vector perturbations decay quickly, so we assume $\delta g_{0,i}\approx 0, \delta g_{0,0}\approx 0$.

Substituting $\phi(\tau,x)=\overline{\phi}(\tau,x)+\frac{f(\tau,x)}{a(\tau)}$ and separating the first-order small quantity, we can obtain the Klein-Gordon equation (this is because the Klein-Gordon equation is obtained by expanding the action to the first order perturbation).

Separating the second-order small quantity:

$$\begin{aligned}S^{(2)}&=\frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x\left[(f')^2-(\nabla f)^2-2\mathscr{H}ff'+(\mathscr{H}^2-a^2V_{,\phi\phi})f^2\right]\\&=\frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x\left[(f')^2-(\nabla f)^2+\left(\frac{a''}{a}-a^2V_{,\phi\phi}\right)f^2\right]\end{aligned}$$

Considering the slow-roll approximation, we obtain the **Mukhanov-Sasaki equation**:

$$\because H^2\approx\frac{V}{3M_{\mathrm{pl}}^2},\ |\eta_{\mathrm{v}}|\equiv M_{\mathrm{pl}}^2\frac{|V''|}{V}\ll 1$$

$$\therefore \frac{V_{,\phi\phi}}{H^2}\approx\frac{3M_{\mathrm{pl}}^2V_{,\phi\phi}}{V}=3\eta_{\mathrm{v}}\ll 1$$

$$\because \text{slow roll makes H approx const}\,,\ a'=a^2H$$

$$\therefore \frac{a''}{a}\approx 2a'H=2a^2H^2\gg a^2V_{,\phi\phi}$$

$$\text{Thus } S^{(2)}\approx\int \mathrm{d}\tau \mathrm{d}^3x\,\frac{1}{2}\left[(f')^2-(\nabla f)^2+\frac{a''}{a}f^2\right]$$

This is the action for the perturbation part of the scalar field.

2.1.1 Quantize perturbation term for the Inflation field

Using the method of classical canonical quantization, we introduce the commutation relation and then quantize the scalar field accordingly.

•First, obtain the canonical variables in momentum representation:

$$\pi\equiv\frac{\partial\mathscr{L}}{\partial f'}=f'$$

•Introduce the commutation relation:

$$[\hat{f}(\tau,x),\hat{\pi}(\tau,x')]=i\delta(x-x')\Rightarrow[\hat{f}_k(\tau),\hat{\pi}_{k'}(\tau)]=i\delta(k+k')$$

•Expand in terms of creation and annihilation operators:

$$\hat{f}_k(\tau) = f_k(\tau)\hat{a}_k + f_k^*(\tau)\hat{a}_k^\dagger$$

•Given the commutation normalization condition:

$$W[f_k, f_k^*] \times [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta(k+k') \Rightarrow W[f_k, f_k^*] = 1$$

•Choose the vacuum state to obtain the Bunch-Davies Vacuum: We define the ground state as the vacuum state. This choice equivalent to choosing the function deep inside the horizon and retard to Minkowski space. And we have Mode Functions:
$\lim_{\tau\to-\infty} f_k(\tau) = \frac{1}{\sqrt{2k}}e^{-ik\tau}$

Solution in de Sitter Space

The previous solution is still quite complex. To further simplify the problem, consider the solution in de Sitter space.

Definition 2.1.1 — de Sitter space time. $\varepsilon \to 0, H \approx const$

$$\text{Proof}\ \frac{a''}{a} = \frac{2}{\tau^2}$$

$$\because d\tau = \frac{dt}{a}$$

$$\tau_f - \tau_i = \int_{\tau_i}^{\tau_f} d\tau = \int_{t_i}^{t_f} e^{-Ht} dt$$

$$\Rightarrow \frac{1}{H}e^{-Ht}|_{t_i}^{t_f} = -\frac{1}{H}[\frac{1}{a_f} - \frac{1}{a_i}]$$

$$\Rightarrow \tau = \frac{1}{aH}$$

Thus, in the de-Sitter background, the **Mukhanov-Sasaki equation** becomes:

$$f_k'' + \left(k^2 - \frac{2}{\tau^2}\right) f_k = 0$$

The solution is:

$$f_k = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right)$$

Now we have completed the process of quantization and obtained the evolution equation in the Heisenberg picture for the canonical variables. Since observable quantities in quantum systems correspond to the inner product of quantum state, let's first calculate the zero-point fluctuation to have a taste of it:

To do this, we perform Wick contractions:

$$\langle|\hat{f}|^2\rangle \equiv \langle 0|\hat{f}^\dagger(\tau,0)\hat{f}(\tau,0)|0\rangle = \int \mathrm{d}\ln k\, \frac{k^3}{2\pi^2}|f_k(\tau)|^2$$

This gives us the power spectrum:

$$\Delta_f^2(k,\tau) \equiv \frac{k^3}{2\pi^2}|f_k(\tau)|^2$$

Now let's go back to $\Delta_{\delta\phi}^2$ by defining $\delta\phi = \frac{f}{a}$:

$$\Delta_{\delta\phi}^2(k,\tau)a^{-2} = \Delta_f^2(k,\tau) = \left(\frac{H}{2\pi}\right)^2\left(1+\left(\frac{k}{aH}\right)^2\right)$$

2.2 Can we directly observe this quantity?

The answer is no, the process is much more complex. Let's first understand what the scalar perturbation describes.

$\delta\phi_k$ corresponds to the component of the scalar perturbation with wavevector k, which can be thought of as a density wave of a scalar field. $\lambda \propto \frac{1}{k}$ corresponds to the wavelength of this density wave in the k-mode. Since Fourier modes cover all modes in momentum space, there will certainly be density waves with very long wavelengths,longer than Horizon's radius. So what happens when some wavelengths become longer than the Hubble horizon?

2.2.1 What happens in the superhorizon regime?

Frozen

A somewhat intuitive but not rigorous statement is that we can consider the scalar field as a continuous elastic medium. When the k-mode is on a super-horizon scale, which corresponds to $\lambda > R_{\text{Hubble}}$, the wave does not complete a causally connected cycle, so its evolution stops after leaving the horizon since the other end of the rope would never receive the shaking information you delivered. (More detailed derivation can be found in Part 3)

Re-entry

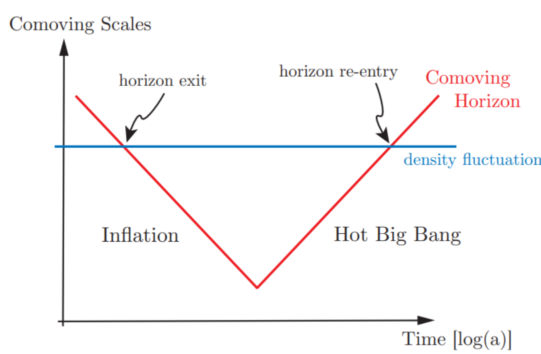


Figure 2.1: Re-entry

The evolution of the Hubble radius during inflation follows the graph shown above. k-modes with wavelengths longer than the Hubble radius will leave the horizon and re-enter the horizon later. They will create perturbations in the cosmic microwave background (CMB) that we can observe(And have contribution in LSS formation). More precisely, what we observe is the perturbation of $\mathscr{R}$ caused by the perturbation of ϕ. The relationship between the two will be explained in Part 3, and the specific observational methods will be described in Part 4. The whole story about observation of inflation is that we observe CMB and LSS which document the evolution after re-entry. Using inverse transfer function that transfer the fluctuation from re-entry to present, and the fact that perturbation mode do not evolve in Sup-horizon scale, we can get the information about initial zero-point fluctuation from the observed data. How can we describe the initial fluctuation? We need to quantize the theory.

Quantize the field

For a scalar field, we have a well-established method to describe it quantum mechanically, called quantization.As for why we can quantize it, one can refer to the "phase space formula–wiki" for a detailed explanation of the transition from phase space to quantization. The reason why we need to think about this question is that we are about to see operator in the QFT will become classic. And I think one would better have a deep thought on the inverse question - Why we can quantize a scaler field. Though it wouldn't stop you from understanding what will be discussed below, I hope you can just think about this essential question.

So we assume that quantization can be performed.

Here's a more intuitive and phenomenological understanding: •When the observation scale reaches the order of , commutation has to be added:$[\hat{x},\hat{p}] = i$which means that the uncertainty principle can be clearly observed. The classical mechanics theory is considered as an "absolute truth" and an approximation at macroscopic scales, i.e., high-energy physics is "renormalized" and the phenomenological explanation is that the influence of the uncertainty principle can be ignored.

•However, this approximation cannot be used when dealing with observations at the microscopic scale. Instead, we need to introduce commutation relations to describe the system. Commutation relations are considered as the fundamental property of this world, allowing us to introduce them more reasonably.

Transition from quantum to classical

Inflation reveals a transition from Quantum to Classical.$\hat{f}$ represents the perturbation of the inflaton field, and we describe the canonical coordinates and canonical momenta of this system as:

$$\hat{f}(\tau,x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[f_k(\tau)\hat{a}_k + f_k^*(\tau)a_k^\dagger\right] e^{ik\cdot x}$$

$$\hat{\pi}(\tau,x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[f_k'(\tau)\hat{a}_k + (f_k^*)'(\tau)a_k^\dagger\right] e^{ik\cdot x}$$

By solving the Mukhanov-Sasaki equation: $f'' + (k^2 - \frac{a''}{a})f = 0$, we can obtain:

$$f_k(\tau) = \alpha \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right)$$

It can be observed that in the super-horizon limit, $k\tau \to 0$,
the solution becomes:

$$f_k(\tau) \approx -\frac{1}{\sqrt{2}k^{3/2}}\frac{i}{\tau} \quad \text{and} \quad f_k'(\tau) \approx \frac{1}{\sqrt{2}k^{3/2}}\frac{i}{\tau^2}$$

Thus, we have $\hat{\pi} = -\frac{1}{\tau}\hat{f}$
Therefore, $[\hat{\pi},\hat{f}] = 0$

Once the commutation relation is lost, there is no longer a canonical quantization scheme, and everything reverts back to classical. The transition from quantum to classical is not a trivial process. By applying the canonical quantization scheme, the action is quantized, and the quantum fluctuations are stretched during inflation into classical perturbations $\mathscr{R}$. It is truly a remarkable phenomenon.

Here, I will share some thoughts on quantization and explore the correspondence between quantum and classical.This transition is amazing, which gaps the from quantum level to cosmology scale. It is hard to keep asking What is canonical quantization, why do we use creation and annihilation operators to represent variables, and is this representation unique? I didn't have the precise answer but I will attach my thought in the Appendix.

2.2.2 Power spectrum

Let's get back to the main topic and continue studying the power spectrum. Since $\delta\phi$ is a perturbation of the scalar field, the scalar field itself is not directly observable. The scalar field determines the energy density, which in turn affects the energy-momentum tensor. The energy-momentum tensor influences the spacetime geometry, leading to changes in $\mathscr{R}$.

For a specific derivation, please refer to Part 3 where we establish the connection between $\mathscr{R}$ and ϕ using the following equation:

$$\mathscr{R} = -\frac{\mathscr{H}}{\bar{\phi}'}\delta\phi \ , \ \varepsilon = \frac{\frac{1}{2}\phi^2}{M_{\mathrm{pl}}^2 H^2}$$

Consequently, the power spectrum can be expressed as:

$$\Delta_{\mathscr{R}}^2(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M_{\mathrm{pl}}^2}\bigg|_{k=aH}$$

In the slow-roll approximation, we have $H^2 \approx \frac{V}{3M_{\mathrm{pl}}^2}$ and $\varepsilon \approx \frac{M_{\mathrm{pl}}^2}{2}\left(\frac{V'}{V}\right)^2$, where $V(\phi)$ represents the potential.

$$\Delta_{\mathscr{R}}^2 = \frac{1}{12\pi^2}\frac{V^3}{M_{\mathrm{pl}}^6 (V')^2}$$

Power spectrum in terms of k:

So far, we have considered the evolution of perturbations during the inflation process. Now let's go back to the initial question: How do we observe inflation? We have to build connection from $\mathscr{R}$with $\overline{T}_{CMB}$ or LSS ,one important tool for this is the power spectrum.

It is important to note that the calculation of $\Delta_{\mathscr{R}}^2$ above was done at $k = aH$. However, this does not mean that $\Delta_{\mathscr{R}}^2$ is independent of k. Since $k = aH$ represents the physical scale at which these modes leave the horizon, different values of k correspond to different times of leaving the horizon, resulting in k-dependence.

We can express $\Delta_{\mathscr{R}}^2(k)$ as a power law in k:

$$\Delta_{\mathscr{R}}^2(k) \equiv A_s\left(\frac{k}{k_\star}\right)^{n_s-1}$$

Expression for n_s:

$$n_s - 1 \equiv \frac{d\ln\Delta^2_{\mathscr{R}}}{d\ln k} = \frac{d\ln\Delta^2_{\mathscr{R}}}{dN} \times \frac{dN}{d\ln k}$$

Since

$$\Delta^2_{\mathscr{R}}(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH}$$

we have:

$$\frac{d\ln\Delta^2_{\mathscr{R}}}{dN} = 2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN} = -2\varepsilon - \eta$$

Considering $k = aH$, we find $\ln k = N + \ln H$. Therefore,

$$\frac{dN}{d\ln k} = \left[1 + \frac{d\ln H}{dN}\right]^{-1} \approx 1 + \varepsilon$$

Taking into account the first-order approximation of the Hubble slow-roll parameter, we obtain:

$$n_s = -2\varepsilon - \eta + 1$$

Alternative derivation of perturbations:

The previous derivation was based on the perturbed action of ϕ to obtain the evolution equation for $\delta\phi$, and then using the relationship between $\delta\mathscr{R}$ and $\delta\phi$ to obtain the evolution equation for perturbations of $\mathscr{R}$, which combined with observables.

The two approaches may seem different, but the astonishing fact is that they lead to the same results. The key is GAUGE!. Read Part 3 for a more comprehensive understanding.

This time, we consider the complete action,

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[R + \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

Due to the fact that all perturbations are related to the perturbation $\mathscr{R}$, we choose the following gauge:

$$\delta\phi = 0,, \qquad g_{ij} = a^2[(1-2R)\delta_{ij} + h_{ij}],, \qquad \partial_i h_{ij} = h^i{}_i = 0$$

Regarding $\Psi \to \mathscr{R}$:

Since

$$\delta\phi \sim \delta q$$

where $\mathscr{R} = \Psi + \frac{H\delta q}{\bar{\rho}+\bar{P}}$, and q corresponds to momentum density, we have

$$\delta\phi = 0 \Rightarrow \mathscr{R} = \Psi$$

The tensor part should be traceless, and the vector part corresponds to

$$\partial_i h_{ij} = h^i_i = 0$$

.

By expanding the action and obtaining the second-order equation for $\mathscr{R}$ (to be further calculated), we introduce the Mukhanov variable:

$$v \equiv z\mathscr{R}$$

where

$$z^2 \equiv a^2 \frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon$$

.

This leads to:

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right]$$

Using a Fourier expansion, we obtain the same Mukhanov-Sasaki Eqn:

$$v_k'' + \left(k^2 - \frac{z''}{z}\right) v_k = 0$$

This result is consistent with the previous perturbative expansion for $\delta\phi$. Despite different approaches, the results remarkably agree, demonstrating the magic of the gauge choice.

The subsequent steps are the same as before.

2.3 Gravitational Waves:

Previously, we considered scalar perturbations. Now let's consider tensor perturbations.We only consider perturbations in space:

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \left(\delta_{ij} + 2\hat{h}_{ij}\right)\mathrm{d}x^i\mathrm{d}x^j\right]$$

The Einstein-Hilbert action is given by:

$$S_{EH} = \frac{{M_{pl}}^2}{2}\int d^4x\sqrt{-g}R$$

Expanding to second order, we obtain the action for tensor perturbations:

$$S^{(2)} = \frac{M_{\mathrm{pl}}^2}{8}\int \mathrm{d}\tau \mathrm{d}^3x\, a^2\left[(\hat{E}_{ij}')^2 - (\nabla\hat{E}_{ij})^2\right]$$

Using the following notation:

$$\frac{M_{\mathrm{pl}}}{2}a\hat{E}_{ij} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} f_+ & f_\times & 0 \\ f_\times & -f_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We obtain:

$$S^{(2)} = \frac{1}{2}\sum_{I=+\times}\int \mathrm{d}\tau \mathrm{d}^3x\left[(f_I')^2 - (\nabla f_I)^2 + \frac{a''}{a}f_I^2\right]$$

We observe that the form of the tensor perturbations is exactly the same as that of the scalar perturbations. So, to some extent, tensor perturbations can be seen as two copies of scalar perturbations.

Therefore, we define the power spectrum of tensor perturbations as follows:

$$\Delta_t^2 \equiv 2 \times \Delta_{\hat{E}}^2 \equiv 2 \times \left(\frac{2}{aM_{\mathrm{pl}}}\right)^2 \times \Delta_f^2$$

Using the relation

$$\Delta_{\delta\phi}^2(k) \approx \left(\frac{H}{2\pi}\right)^2 \bigg|_{k=aH}$$

we obtain

$$\Delta_t^2(k) = \frac{2}{\pi^2} \frac{H^2}{M_{\mathrm{pl}}^2} \bigg|_{k=aH}$$

Comparing with the perturbation of $\mathscr{R}$:

$$\Delta_{\mathscr{R}}^2(k) = \frac{1}{8\pi^2} \frac{1}{\varepsilon} \frac{H^2}{M_{\mathrm{pl}}^2} \bigg|_{k=aH}$$

We can see that the tensor perturbations do not have the factor ε. Therefore, if we observe perturbations in Δ_t^2, we can directly measure information about the inflation field potential.

3. Perturbation

Earlier time, we mentioned perturbations, but we did not delve into the evolution of perturbations. Our idea was that if $\phi \to \bar{\phi} + \delta\phi$, it would lead to $\rho \to \bar{\rho} + \delta\rho$ and generate $\delta\mathscr{R}$. But how do these things evolve?

3.1 How perturbation evolve with universe expansion? (Newtonian Version)

Newtonian perturbation

Let's have some intuitive insights into the evolution of Adiabatic perturbations. To start, we can consider the matter, radiation, and other substances in the universe as ideal fluids. We will discuss the perturbations within the framework of Newtonian mechanics and explore how they evolve as the universe expands.

Definition 3.1.1 — Adiabatic perturbation. This type of perturbation refers to the local state of matter at a spacetime point (τ, x) being the same as the background universe at a different time $\tau + \delta\tau(x)$.The properties of perturbations apply to all types of matter, given by $\frac{\delta_I}{1+w_I} = \frac{\delta_J}{1+w_J}$.where $\delta$is the density contrast.

Definition 3.1.2 — Isocurvature perturbations . This is a complement to adiabatic perturbations.

Adiabatic perturbations correspond to changes in the total density, while isocurvature perturbations correspond to mixing between different types of perturbations.$S_{IJ} \equiv \frac{\delta_I}{1+w_I} - \frac{\delta_J}{1+w_J}$ In the language of single-field inflation, the primordial perturbations are adiabatic, meaning that $S_{IJ} = 0$.

We can apply this kinds of perturbation only in suitable cases:Non-relastivistic+sub-Hubble range(When we derive the scenario of an expanding universe, we employ low-order approximations, thus imposing requirements on the expansion order which requires sub-horizon discussion .)There are two equation governs the dynamic of fluid: Continuity **eqn + Euler eqn**

•Continuity equation:

$$[\frac{\partial}{\partial t} - Hx\cdot\nabla][\bar{\rho}(1+\delta)] + \frac{1}{a}\nabla\cdot[\bar{\rho}(1+\delta)(Hax+v)] = 0$$

Zeroth-order approximation:

$$\frac{\partial\bar{\rho}}{\partial t} + 3H\bar{\rho} = 0$$

First-order approximation derived from the zeroth-order result:

$$\dot{\rho} = -\frac{1}{a}\nabla\cdot v$$

Note:

Zeroth-order approximation and the first-order approximation are independent allows us to substitute the zeroth-order result into the first-order approximation. In the first-order approximation, the x term is dropped because x is a small quantity. This is why the calculation can only be done within the sub-Hubble radius.

•Euler Equation:

$$\dot{v} + Hv = -\frac{1}{a\bar{\rho}}\nabla\delta P - \frac{1}{a}\nabla\delta\Phi$$

•Poisson Equation for gravitational potential:

$$\nabla^2\delta\Phi = 4\pi Ga^2\bar{\rho}\delta$$

Evolution Equation and Jeans' Instability

Since Fluid-Dynamic eqns and the source term eqn are settled, we get evolution equation of the perturbation by directly combining the above three equations

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta = 4\pi G\bar{\rho}\delta$$

Assuming the general solution as $\delta = Ae^{i(wt-k\cdot r)}$, we substitute it into the equation. By solving the discriminant for $w = 0$, the corresponding k is the Jeans wave number.

$$\frac{c_s^2}{a^2}k^2 = 4\pi G\bar{\rho}$$

The discriminant is given by $-w^2 + 2iHw + c_s^2k^2 = 4\pi G\bar{\rho}$, which represents a quadratic curve in terms of w. The solutions for w involve imaginary numbers, indicating that the perturbation of δ expands or decays with time.

$w = i(H \pm \sqrt{H^2 - (4\pi G\bar{\rho} - c_s^2k^2)})$ is the solution to the quadratic equation. Since we approximate the inequality as the above equation, we choose the - sign. Hence, for perturbations larger than the Jeans wavelength, the perturbation exponentially expands, while for perturbations smaller than the Jeans wavelength, the perturbation exponentially decays.

Definition 3.1.3 — Jeans' Instability and wave number:. Jeans' Instability means where is a critical point where the wave has $w \equiv 0$ corresponding to stop evolving. And we'll have the wave number. After calculation we get two modes, one for Amplitude decay, another one is increasing. Neither of those state are stable. And that's what instability means.

3.2 How perturbation evolve with universe expansion? (Relativistic Version)

Due to the description of cosmic evolution in general relativity, a more accurate perturbation theory needs to be considered in the relativistic case. Since relativity involves solving a set of multivariable nonlinear differential equations, believe me, it can be extremely complex to solve. If you have forgotten that feeling about how hard it might be, just recall the agony of deriving the friedmann equations. Trust me, it's quite challenging. Therefore, I won't go through the complete derivation of relativistic perturbations here.

However, if you're curious, you can refer to Baumann's lecture notes [Bau22]. Just flipping through Chapter 4 will leave you dazzled by the overwhelming symbols. In my point of view, a more concise approach is to remember the general idea behind the derivation and be aware of some key results. Our discussion focuses on inflation, and the previous pitfall primarily involved the relationship between $\mathscr{R}$ and $\delta\phi$. By knowing our main emphasis, we won't get lost in the sea of mathematics. If you're prepared, then let's set sail.

3.2.1 Perturbed spacetime

There are various types of perturbations: 1. Perturbations to spacetime, 2. Perturbations to matter, 3. Perturbations to curvature.When we consider linearized perturbation theory, we only need to take care of those scalers and leave vector and tensor behind. We wouldn't proof it here that vector mode is negligible during inflation since those mode would decay rapidly

General form of perturbation on metrics

When we talk about perturbations in the context of general relativity, what we need to consider is the perturbation of the Ricci scalar or the tensor part. Since the metric gives us the connection and, upon contraction, yields the Ricci scalar or you can have some tensor perturbation. We generally write the perturbation in terms of the perturbed metric

$$ds^2 = a^2(\tau)[(1+2A)d^2\tau - 2B_i dx^i d\tau - (\delta_{ij} + h_{ij})dx^i dx^j]$$

Recalling that claimed by Helmholtz that any vector can be decomposed into two different part, this Metric perturbation can be decomposed into different part as well. The mathematical detail wouldn't be discuss here, and more discussion over SVT would be seen in Appendix.

Definition 3.2.1 — Scaler-Vector-Tensor decomposition. The decomposition states that the evolution equations for the most general linearized perturbations of the friedmann–Lemaître–Robertson–Walker metric can be decomposed into four scalars, two divergence-free spatial vector fields (that is, with a spatial index running from 1 to 3), and a traceless, symmetric spatial tensor field with vanishing doubly and singly longitudinal components. –wiki

$$\begin{cases} B_i = \partial_i B + \hat{B}_i \\ h_{ij} = 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{E}_{ij} \\ \partial_{\langle i}\partial_{j\rangle}E \equiv (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E \\ \partial_{(i}\hat{E}_{j)} \equiv \frac{1}{2}(\partial_i\hat{E}_j + \partial_j\hat{E}_i) \end{cases}$$

Here, we have 4 scalars A, B, CE ,things like $\partial_i B$ means curl-free, 2 vectors $\hat{B}, \hat{E}$, and 1 tensor E_{IJ}. To count the degrees of freedom (DOF), we can go back to ds^2 and write it in the form of a symmetric matrix, where we find DOF = 10. However, the four space-time coordinates can be arbitrarily chosen, representing 4 gauge degrees of freedom. **This is where the gauge stem from!**

Be aware of gauge

•Fictitious perturbation:

If different reference frames are chosen, for example, a spatial perturbation $x^i \to x^i + \xi^i(\tau, x) \equiv \tilde{x}^i$, then the after-transformation metric also has the form of metric perturbation, but it is a fictitious perturbation.

$$ds^2 = a^2(\tau)[d^2\tau - 2\xi_i' d\tilde{x}^i d\tau - (\delta_{ij} + 2\partial_{(i}\xi_{j)})d\tilde{x}^i d\tilde{x}^j]$$

•Eat up perturbation:

Vice versa,if we have a perturbation, it can be hid with specific choosing of gauge. $\rho(\chi, \tau)$change continuously in the space-time, we can always find a 3-sphere that makes ρ stays constant on the sphere.

Gauge is a serious Problem. If we want to settle the problem, we have to consider both the gauge in Matter perturbation and perturbation in Space-time.

3.2.2 Gauge Problem

Coordinate Transformation

Let's consider the coordinate transformation as follows,which differs time and spacial coordinate by mixing 3 spatial coordinate and allow us to decompose it into curl-free part and div-free part. This kind of transformation represents one of the most general form of coordinate transformation.

$$\begin{cases} X^\mu \to \tilde{X}^\mu \equiv X^\mu + \xi^\mu(\tau, x) \\ \xi^0 \equiv T \\ \xi^i \equiv L^i = \partial^i L + \hat{L}^i \end{cases}$$

This describes the translation in each dimension, with time being independent, so there is only scalar translation. The three spatial dimensions are coupled, and using the SVT decomposition, they can be decomposed into divergence-free and curl-free translation components.

Perturbation Transformation:

By substituting the coordinate translations into the metric, we can obtain the transformation equations for the perturbation parameters $ABCE\ldots$.

Let's use an example to show those Transformation is valid:

$$\begin{aligned} a^2(\tau)\big(1+2A\big) &= \big(1+T'\big)^2 a^2(\tau+T)\big(1+2\tilde{A}\big) \\ &= \big(1+2T'+\cdots\big)\big(a(\tau)+a'T+\cdots\big)^2\big(1+2\tilde{A}\big) \\ &= a^2(\tau)\big(1+2\mathscr{H}T+2T'+2\tilde{A}+\cdots\big) \end{aligned}$$

Under first order approximation we have will get the Transformation of A and same procedure can be put on other variables.

$$\begin{cases} A \to A - T' - \mathscr{H}T \\ B \to B + T - L' \\ C \to C - \mathscr{H}T - \dfrac{1}{3}\nabla^2 L \\ \hat{B}_i \to \hat{B}_i - \hat{L'}_i \\ \hat{E}_i \to \hat{E}_i - \hat{L'}_i \\ \hat{E_{ij}} \to \hat{E_{ij}} \end{cases}$$

Above all are gauge transformation. There's a natural question that one may want to ask : Is there any variable that is consistent before and after the transformation?

Yes they are **Bardeen Variable:**Those invariants are constructed from the transformation relations of perturbations.

Definition 3.2.2 — Bardeen Variable.

$$\begin{cases} \Psi \equiv A + \mathscr{H}(B - E') + (B - E')' \\ \hat{\Psi} \equiv \hat{E}_i' - \hat{B}_i \\ \hat{E}_{ij} \\ \Phi \equiv -C - \mathscr{H}(B - E') + \dfrac{1}{3}\nabla^2 E \end{cases}$$

It can easily to verify that the Bardeen variables remain invariant under coordinate transformations. Therefore, these quantities are gauge invariant.

Therefore, genuine perturbations in spacetime should be expressed in terms of perturbations in the Bardeen variables. What we call genuine means GR allowed, which satisfy basic mathematical rules in GR and the Equivalence principle!

Note:

This equation describes a translation in spacetime, and at the same time, the corresponding metric has a variation. So please do not forget $a(\tau) \to a(\tau + T)$ when trying to deriving upper relations.

The perturbation transformation is obtained by expanding each element of the matrix $g_{\mu\nu}(X) = \frac{\partial \tilde{X}^\alpha}{\partial X^\mu}\frac{\partial \tilde{X}^\beta}{\partial X^\nu}\tilde{g}_{\alpha\beta}(\tilde{X})$, and retaining the terms up to first order in small quantities.

3.2.3 Gauge Fixing:

Since gauge degrees of freedom correspond to the choice of four coordinates, four constraint conditions are needed to eliminate the gauge degrees of freedom.Instead of using invariant as the representation of variable, we can choose a specific gauge to eliminate redundance freedom. Well I can't give a rigorious proof here however the main concept lying beneath is DOF will decide all the evolution. Consider this kind of gauge a convention in transforming vector field between two different manifolds, all we need to do is to eliminate the extra degree of freedom thus those two convention will lead us to same result. Let's come back to the gauge there are some commonly seen gauge as follows

•Newtonian Gauge:

$$\begin{cases} B = E = 0 \quad \text{(gauge condition)} \\ ds^2 = a^2(\tau)[(1 + 2\Psi)d^2\tau - (1 - 2\Phi)\delta_{ij}dx^i dx^j] \quad \text{(corresponding metric)} \end{cases}$$

Since $B=0$, $B_i=0$ corresponds to three constraint conditions in the metric matrix; $E=0$ is the fourth constraint condition.

By substituting $B=E=0$, we have Ψ and Φ identified as A and $-C$ respectively.

Spatially Flat Gauge: obviously all variation in spatial coordinate are 0.

$$\begin{cases} C=0 \\ E=0 \end{cases}$$

3.2.4 Perturbed Matter:

According to the Einstein equations, perturbations to the stress-energy tensor will induce perturbation in metric. Or equivalently saying that metric perturbation would ultimately and uniquely determine stress-energy perturbation. We can eliminate extra DOF through gauge fixing terms in stress-energy tensor perturbation as an alternative in fixing terms in metric perturbation.

1.Stress Energy Tensor and its perturbation

$$\begin{cases} T^{\mu}_{\nu}=(\bar{\rho}+\bar{P})\bar{U}\mu\bar{U}_{\nu}-\bar{P}\delta^{\mu}_{\nu} \\ \delta T^{\mu}_{\nu}=(\delta\rho+\delta P)\bar{U}\mu\bar{U}_{\nu}+(\bar{\rho}+\bar{P})(\delta U^{\mu}\bar{U}_{\nu}+\bar{U}^{\mu}\delta U_{\nu})-\delta P\delta^{\mu}_{\nu}-\Pi^{\mu}_{\nu} \end{cases}$$

Note: the anisotropic term Π^{μ}_{ν} can be eliminated and detailed discussion would be found in Appendix.

Definition 3.2.3 — momentum density q**.** The definition of q is:

$$q^i \equiv (\overline{\rho}+\overline{P})v^i$$

According to SVT it can be separate to two part

$$q_i=\partial_i q+\hat{q}_i$$

•$T^{\mu}{}_{\nu}$'s component's perturbation

$$\begin{cases} \delta T^{0}{}_{0}=\delta\rho \\ \delta T^{i}{}_{0}=(\bar{\rho}+\bar{\mathscr{P}})v^i \\ \delta T^{0}{}_{j}=-(\bar{\rho}+\bar{\mathscr{P}})(v_j+B_j) \\ \delta T^{i}{}_{j}=-\delta P\delta^i_i-\Pi^{i}{}_{j} \end{cases}$$

2.Gauge Issues of Perturbations in Energy-Momentum Density

Perturbations can be applied to the energy-momentum density, but they take different forms in different coordinate systems.We'll use the coordinate transformation mentioned in metric perturbtion.Therefore, it is necessary to discuss the gauge issues of the energy-momentum density.

Transformation forms of energy-momentum density in different reference frames:

$$T^{\mu}{}_{\nu}(X)=\frac{\partial X^{\mu}}{\partial\tilde{X}^{\alpha}}\frac{\partial\tilde{X}^{\beta}}{\partial X^{\nu}}\tilde{T}^{\alpha}{}_{\beta}(\tilde{X})$$

$$\begin{cases} \delta\rho \mapsto \delta\rho - T\bar{\rho}' \\ \delta P \mapsto \delta P - T\bar{P}' \\ q_i \mapsto q_i + (\overline{\rho} + \overline{\mathscr{P}})L_i' \\ v_i \mapsto v_i + L_i' \\ \Pi_{ij} \mapsto \Pi_{ij} \end{cases}$$

When dealing with energy-momentum in different reference frames, there are two approaches: 1. Finding gauge invariants, and 2. Choosing a gauge.

Gauge Invariants:

$$\bar{\rho}\Delta \equiv \delta\rho + \bar{\rho}'(v+B)$$

Here, the definition of v is $\partial_i v = v_i$. Δ is called the "comoving-gauge density perturbation."

Gauges:

$*$Uniform density gauge: No perturbation happens in energy density

$$\delta\rho = 0$$

$*$Comoving gauge: observer can't see the "flux"

$$q = 0$$

3. Linearized Evolution Equation:

By choosing the Newtonian gauge, we can obtain $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$ mentioned above, from which we can determine the perturbed $g_{\mu\nu}$ and $T_{\mu\nu}$.Due to the condition required by General Relativity ($\nabla_\mu T^\mu{}_\nu = 0$), we can use this equation to relate $\delta g_{\mu\nu}$ and $\delta T_{\mu\nu}$ and derive the evolution equation. The detailed steps wouldn't be demonstrate here, what we will show is key steps in derivation and some important result.

$$\begin{cases} \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \\ g_{\mu\nu} = a^2 \begin{pmatrix} 1+2\Psi & 0 \\ 0 & -(1-2\Phi)\delta_{ij} \end{pmatrix} \\ \nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\mu\nu} T^\nu{}_\alpha \end{cases}$$

•Simplifying the Relativistic Continuity Equation by setting $\nu = 0$

$$\bar{\rho}' + \delta\rho' + \partial_i q^i + 3H(\bar{\rho} + \delta\rho) - 3\bar{\rho}\Phi' + 3H(\bar{P} + \delta P) - 3\bar{P}\Phi' = 0$$

Zeroth-order term:

$$\overline{\rho}' = -3\mathscr{H}(\overline{\rho} + \overline{P})$$

This equation implies energy conservation in a homogeneous background; it can be understood that $\rho' \sim -3\mathscr{H}\rho$ corresponds to dilution, while $\bar{P}$ corresponds to work done on the surroundings.

First-order perturbation term:

$$\delta\rho' = -3\mathscr{H}(\delta\rho + \delta P) + 3\Psi'(\overline{\rho} + \overline{P}) - \nabla \cdot q$$

The first term in this equation, like the zeroth-order term, arises from dilution caused by cosmic expansion. The last term represents local fluid flow due to peculiar velocities. As for the middle term, the Φ' component originates from the perturbation effects of the local expansion rate associated with the perturbed metric field $a(1-\Phi)$.

Define: $\delta \equiv \frac{\delta\rho}{\rho}; v \equiv \frac{q}{\bar{\rho}+\bar{P}}$

Relativistic Continuity Equation:

$$\delta' + \left(1+\frac{\bar{P}}{\bar{\rho}}\right)\left(\nabla\cdot v - 3\Phi'\right) + 3\mathscr{H}\left(\frac{\delta P}{\delta\rho} - \frac{\bar{P}}{\bar{\rho}}\right)\delta = 0$$

Relativistic Euler Equation:

setting $v = i$ we have:

$$v' + \mathscr{H}v - 3\mathscr{H}\frac{\bar{P}'}{\bar{\rho}'}v = -\frac{\nabla\delta P}{\bar{\rho}+\bar{P}} - \nabla\Psi$$

Perturb Einstein Equation:

$$\begin{cases} G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \\ G_{00} = 3\mathscr{H}^2 + 2\nabla^2\Psi - 6\mathscr{H}\Psi' \\ G_{0i} = 2\partial_i(\Psi' + \mathscr{H}\Phi) \\ G_{ij} = -(2\dot{\mathscr{H}}' + \dot{\mathscr{H}}^2)\delta_{ij} + \left[\nabla^2(\Psi-\Phi) + 2\Phi'' + 2(2\dot{\mathscr{H}}' + \dot{\mathscr{H}}^2) + \right. \\ \qquad (\Phi+\Psi) + 2\mathscr{H}\Psi' + 4\bar{\mathscr{H}}\Phi'\delta_{ij} + \partial_i\partial_i(\Phi-\Psi) \end{cases}$$

4.Summary

Some conclusions regarding perturbations in the Einstein equations:

1). If we assume that $T^i{}_j$ does not have anisotropic stress, then we can prove that $\partial_{\langle i}\partial_{j\rangle}(\Phi-\Psi) = 0$. Therefore, we can conclude that:

$$\Psi = \Phi$$

$G_{00} = 8\pi GT_{00} \Rightarrow$

$$\nabla^2\Phi = 4\pi Ga^2\bar{\rho}\delta + 3H(\Phi' + \mathscr{H}\Phi)$$

$G_{0i} = 8\pi Gg_{0\mu}T_i^\mu \Rightarrow$

$$\Phi' + \mathscr{H}\Phi = -4\pi Ga^2(\bar{\rho} + \bar{\mathscr{P}})v$$

$G^i{}_i = 8\pi GT_i^i \Rightarrow$

$$\Phi'' + 3\mathscr{H}\Phi' + (2\mathscr{H}' + \mathscr{H}^2)\Phi = 4\pi Ga^2\delta\mathscr{P}$$

Combining first 2 formula

$$\nabla^2\Phi = 4\pi Ga^2\bar{\rho}\Delta$$

Where Δ is defined as: $\bar{\rho}\Delta \equiv \bar{\rho}\delta - 3H(\bar{\rho}+\bar{P})v$.

Note:

$*$Since the Einstein tensor can derive many equations, but due to the existence of the Bianchi identity, many equations are consistent. $*$If you feel puzzled about all those formula Its ok to skip those math and jump into next section.

3.2.5 Comoving Curvature Perturbation:

All we need to consider is the scaler perturbation. Noe let's first considering the intrinsic curvature on a constant-time hypersurface in any gauge for scaler perturbation. Induced metric $\gamma_{ij} \equiv a^2[(1+2C)\delta_{ij}+2E_{ij}]$ Claim that after a very long calculation, we will have comoving curvature perturbation. Please check baumann's lecutre note if you want to know any detail.

$$R = C - \frac{1}{3}\nabla^2 E + \mathscr{H}(B+v)$$

Note
This should be used under the condition of both Comoving and Newtonian Gauges being valid.

Theorem 3.2.1 — R conservation. R is conserved on large scales and for adiabatic perturbations.

Under Newtonian Gauge: $B = E = 0, C = -\Phi$

$$R = -\Phi + \mathscr{H}v$$

Substituting v gives the evolution equation:

$$-4\pi G a^2(\bar{\rho}+\bar{P})\mathscr{R}' \;=\; 4\pi G a^2 \mathscr{H}\delta P_{\text{nad}} + \mathscr{H}\frac{\bar{P}'}{\bar{\rho}'}\nabla^2\Phi$$

For non-adiabatic pressure δP_{n-ad}, if it satisfies the barotropic equation $P \equiv P(\rho)$, and $\mathscr{H}k^2\Psi = \mathscr{H}k^2 R$, it can be solved as:

$$\frac{d\ln R}{d\ln a} \sim \left(\frac{k}{\mathscr{H}}\right)^2$$

For super-Hubble scales: $k << \mathscr{H}$ implies that R does not evolve with time.

Useful formula

•Einstein's Perturbation Equations

$$\begin{cases} \nabla^2\Phi - 3\mathscr{H}(\Phi' + \mathscr{H}\Phi) = 4\pi G a^2\delta\rho \\ \Phi' + \mathscr{H}' + \mathscr{H}'\Phi = 4\pi G a^2 \delta\mathscr{P} \\ \Phi'' + 3\mathscr{H}\Phi' + (2\mathscr{H}' + \mathscr{H}^2)\Phi = 4\pi G a^2 \delta P \end{cases}$$

•Comoving Curvature Perturbation:

$$\mathscr{R} = -\Phi - \frac{\mathscr{H}(\Phi' + \mathscr{H}\Phi)}{4\pi G a^2(\bar{\rho}+\bar{P})}$$

•Relativistic Continuity Equation and Euler Equation induced by the Energy-Momentum Tensor:

$$\delta' + 3H\left(\frac{\delta P}{\delta\rho} - \frac{\overline{P}}{\overline{\rho}}\right)\delta \;=\; -\left(1+\frac{\overline{P}}{\overline{\rho}}\right)(\nabla\cdot v - 3\Phi')$$

$$v' + 3H\left(\frac{1}{3} - \frac{\bar{P}'}{\bar{\rho}'}\right)v = -\frac{\nabla \delta P}{\bar{\rho} + \bar{P}} - \nabla\Phi$$

3.3 Theoretically discuss Perturbation's evolution

3.3.1 Gravitational potential evolution

If there are alot of detailed you didn't understand yet, it is ok. All discussions in the following chapter are based on this equation which is just a simple combination of 3 Einstein perturbation eqn.

$$\Phi'' + 3(1+w)\mathscr{H}\Phi' + wk^2\Phi = 0$$

Since most of the modes would exit the horizon, we'll mainly focus on super-horizon scale $k << \mathscr{H}$ perturbation solution.

$$\Phi'' + 3(1+w)\mathscr{H}\Phi' = 0$$

•Solution 1: $\Phi = const$
•Solution 2: $\Phi = C - \frac{e^{-3(1+w)\mathscr{H}\tau}}{3(1+w)\mathscr{H}}$
We consider the first case as the growing mode and the second case as the decay mode. This solution implies that if the gravitational field still exist in the Super-horizon scale, than it must be frozen

3.3.2 density contrast evolution

Using friedmann's second equation and Einstein's perturbation equation we can derive density evolution equation. We implicitly using the Newtonian gauge and replace $\nabla^2 \to k^2$.

$$\delta = -\frac{2}{3}\frac{k^2}{\mathscr{H}}\Phi - \frac{2}{\mathscr{H}}\Phi' - 2\Phi \Leftarrow \begin{cases} \frac{3}{2}\mathscr{H}^2 = 4\pi G a^2 \bar{\rho} \\ \nabla^2\Phi - 3\mathscr{H}(\Phi' + \mathscr{H}\Phi) = 4\pi G a^2 \delta\rho \end{cases}$$

Obtaining the

$$\delta = -\frac{2}{3}\frac{k^2}{\mathscr{H}}\Phi - \frac{2}{\mathscr{H}}\Phi' - 2\Phi$$

The first term can be neglected when discussing the super-hubble scale.The second term is also small since $\Phi \approx const$ and can be neglected.

Therefore, we can conclude:

Theorem 3.3.1 — Density perturbation outside Hubble radius. Outside the Hubble radius+when the equation of state is constant, the density perturbation is proportional to the perturbation caused by the inflation-induced curvature perturbation.

$$\delta \approx -2\Phi$$

this means density perturbation is approximate frozen as well

Note:

If the equation of state is not constant something interesting is going to happen: We 'll see a huge drop in gravitational potential.

Consider the transition from radiation to matter. We can obtain the expression for $\mathscr{R}$ to express:

$$\mathscr{R} = -\frac{5+3w}{3+3w}\Phi$$

This eqn stems from $\mathscr{R} = -\Phi - \frac{\mathscr{H}(\Phi'+\mathscr{H}\Phi)}{4\pi G a^2(\overline{\rho}+\overline{\mathscr{P}})}$

By conservation of $\mathscr{R}$ we have

$$\mathscr{R} = -\frac{3}{2}\Phi_{\mathrm{RD}} = -\frac{5}{3}\Phi_{\mathrm{MD}}$$

which implies gravitational potential have a huge drop (10%)in the transition.

3.4 Evolution of Perturbations in different scenerio

From above we can tell that the behaviour of perturbation evolution inside/outside horizon and in different era(Radiation or matter) behave differently.Thus We can separate our discussion in four sector.

$$\Phi'' + 3(1+w)\mathscr{H}\Phi' + wk^2\Phi = 0$$

$$\delta = -\frac{2}{3}\frac{k^2}{\mathscr{H}}\Phi - \frac{2}{\mathscr{H}}\Phi' - 2\Phi$$

3.4.1 Gravitational Potential Evolution

Radiation Era

$\mathscr{H}^{-1} \approx \tau$, we can obtain $\Phi'' + \frac{4}{\tau}\Phi' + \frac{k^2}{3}\Phi = 0$ in the Radiation Era. Considering the perturbations in the early universe as Bessel functions, the evolution equation for the gravitational potential in the radiation-dominated period is given by

$$\Phi_k(\tau) = -2R_k(0)\left(\frac{\sin x - x\cos x}{x^3}\right)$$

where $x = \frac{1}{\sqrt{3}k\tau}$.

•Case 1: Outside Horizon $x << 1$

$\lambda \propto k^{-1}$, so this condition implies that λ is sufficiently large. But what is the physical significance of this wavelength? Is this k obtained from the plane wave of the gravitational potential really a propagating wave? What is a propagating wave?

$$\Phi = const$$

Hint: Expanding $sinx - xcosx \sim -\frac{x^3}{2!}$ by Taylor series.

•**Case 2: Inside Horizon** $x >> 1$

$$\Phi_k(\tau) \approx -6R_k(0)\frac{\cos\left(\frac{1}{\sqrt{3}}k\tau\right)}{(k\tau)^2}$$

We can see that the gravitational potential perturbation inside the Horizon oscillates with a frequency of $\frac{k}{\sqrt{3}}$. Due to the previous assumption that there is only one dominant wave, what is obtained here is the relationship between the wave vector and the wave speed of a monochromatic wave. Since the frequency is like this, $v = \frac{c}{\sqrt{3}}$.

Matter Era

Since $w = 0$, there is no longer an influence of k which means it is scale invariant, so it is the same whether it is inside or outside the Horizon. A second-order constant coefficient differential equation has two solutions, corresponding to ϕ being a constant or decaying rapidly to 0.

$$\Phi'' + \frac{6}{\tau}\Phi' = 0 \Rightarrow \Phi \propto \begin{cases} const. \\ \tau^{-5} \propto a^{-5/2} \end{cases}$$

Summary

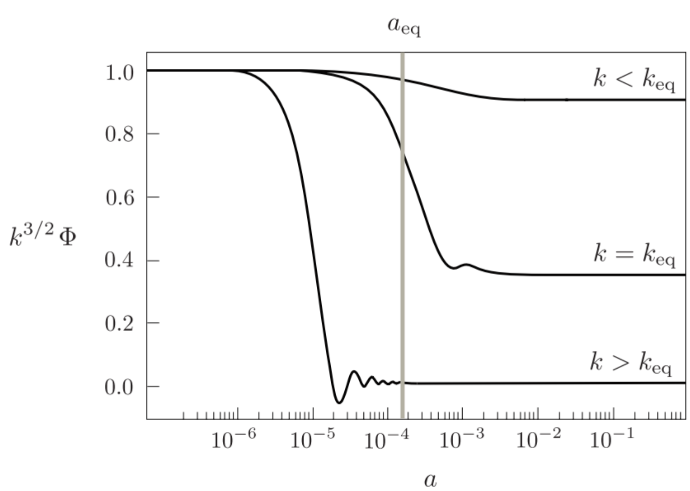


Figure 3.1: Potential suppression

The y-index $k^{3/2}\Phi$ is just a convention. Given k_0 this $k^{3/2}\Phi$ is proportion to Φ so we don not need to care much about it. This diagram describes different reaction for different mode of POTENTIAL perturbation enters the horizon in different era. Top one means entering the Horizon in Matter Dominated Era corresponding to the Φ 10% drop we calculate above.

Others enter the Horizon at radiation dominated era and evolve under the bessel eqn.

3.4.2 Evolution of density contrast

After considering the Potential perturbation we can consider the Density contrast evolution. Something has to mention here. When we are calculating gravitational potential, we do not need to consider about w since potential do not have an equation of state. However, when we are dealing with material, there are more than one equation of state get involved. The material we are looking at and the background dominated material. Which w shall we use? We have to think about how did we get w in their evolution eqn which means we have to trace back to Einstein's Perturbation Equations where the w stem from the term $\bar{\rho}+\bar{P}$. Thus this w correspond to the back groud eqn of state.

•Radiation Era

Let's first consider the evolution of density contrast in the radiation era, which requires the use of the second evolution equation, with$(w=\frac{1}{3})$.

Beyond Horizon

$$\delta_r = -\frac{2}{3}(k\tau)^2\Phi - 2\tau\Phi' - 2\Phi$$

Inside Horizon

$$\Delta_r = -\frac{2}{3}(k\tau)^2\Phi$$

By substituting the evolution equation for the gravitational potential inside the horizon into the equation for density contrast inside the horizon, we obtain $\delta_r = 4\mathscr{R}(0)\cos(\frac{1}{\sqrt{3}k\tau})$. Therefore, in the radiation era, the oscillations inside the horizon have property as follows: 1. Constant amplitude, 2. Constant frequency, and 3. Similar to harmonic oscillations: We put Φ back to δ_r and the solution follows $\delta_r'' - \frac{1}{3}\nabla^2\delta_r = 0$

This equation can be derived from the continuity equation and Euler equation induced by the energy-momentum tensor, and the result is consistent with solving the second-order differential equation.

•Matter Era

Directly substituting the relativistic continuity equation and the relativistic flow conservation equation.We have

$$\begin{cases} \delta_r' = -\frac{4}{3}\nabla\cdot v_r \\ v_r' = -\frac{1}{4}\nabla\delta_r - \nabla\Phi \end{cases}$$

Finally, we obtain the differential equation:

$$\delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi$$

Summary

Radiation dominated era density perturbation oscillate around $\delta_r = 0$

$$\delta_r'' - \frac{1}{3}\nabla^2\delta_r = 0$$

Matter dominated era density perturbation oscillate around $\delta_r = -(-4\Phi_{MD}(k)$

$$\delta_r'' - \frac{1}{3}\nabla^2\delta_r = \frac{4}{3}\nabla^2\Phi$$

4. Observation

When we look up at the universe, the sparkling stars in the sky are mesmerizing. This scene is perfect for holding a glass of red wine, sitting on the green grass with your loved one, whispering and enjoying this romantic and magical moment. However, as a professional physicist, we want to see more than just romance. We want to see data!

When discussing issues on the cosmic scale, we have two main sources of data: large-scale structure (LSS) and the cosmic microwave background (CMB). The former refers to the intricate structures left by the Creator in the vast universe, while the latter represents the lingering glow of the fiery aftermath spread throughout the cosmos.

This chapter we'll introduce some observation in the cosmology. We can use the temperature fluctuation and polarization of photons in CMB and LSS to acquire raw data from the universe. Meanwhile we can use Lyth bound and Coherent phase to futher determine our inflation model.

4.1 Observable Quantities of CMB:

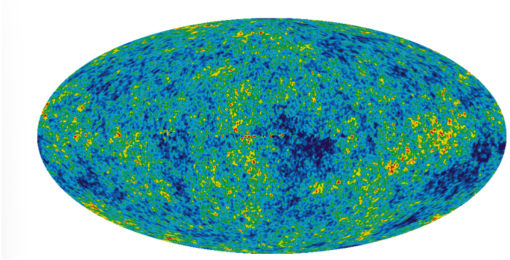


Figure 4.1: CMB

Quantum fluctuations in the field will cause an uneven distribution of density (ρ) in the early universe, resulting in a corresponding in-homogeneous distribution of gravitational potential. Thus, the photons of the CMB also have different behaviour when they climbed out of the gravitational potential. The regions with energy loss for photons correspond to the blue areas in the CMB picture as follow. Therefore, temperature fluctuations directly reflect the fluctuation properties of the early universe's quantum field, or more precisely, the scalar perturbations of the early universe.

We use $\Theta(\hat{n})$ to indicate direction and we can draw the temperature fluctuations dependent on direction. The blue regions represent colder areas where photons lose energy during the process of climbing out of the gravitational potential.

4.1.1 Processing the observing data

The temperature fluctuation can be represented in the following form

$$\Theta(\hat{n}) \equiv \frac{\Delta T(\hat{n})}{T_0}$$

For convenience, we perform a three-dimensional Fourier expansion on it and we have

$$\Theta(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$$

$$a_{lm} = \int \mathrm{d}\Omega Y^*_{lm}(\hat{n})\Theta(\hat{n})$$

Thus we have the angular spectrum of the temperature fluctuation.

4.1.2 Correlation Function in CMB

Second-order correlation function

Here, l corresponds to quantum number for θ, and m corresponds to the quantum number for ϕ. To obtain a rotationally-invariant angular power spectrum, we need to sum over m to obtain C_l^{TT}:

$$C_\ell^{TT} = \frac{1}{2\ell+1}\sum_m \langle a^*_{\ell m} a_{\ell m}\rangle$$

After processing those observable what we need to do is to combine these quantities with physics in early universe.We can formally consider the relationship between the scalar perturbations in the early universe and the observed CMB perturbation spectrum as follows:

$$a_{\ell m} \equiv 4\pi(-i)^\ell \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Delta_{T\ell}(k)\, \mathscr{P}_{\mathbf{k}}\, Y_{\ell m}(\hat{\mathbf{k}})$$

Here, $\Delta_{T\ell}(k)$ represents the transfer function, which will be defined in detail later.

Since $\sum\limits_{m=-\ell}^{m=\ell} Y_{\ell m}(\hat{\mathbf{k}})Y_{\ell m}(\hat{\mathbf{k}}') = \frac{2\ell+1}{4\pi}P_\ell(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')$This allows us to correlate the observable quantities with scalar perturbations:

$$C_\ell^{TT} = \frac{2}{\pi}\int k^2 \mathrm{d}k\; \mathscr{P}_R(k)\Delta^2_{T\ell}(k)$$

If we have the transfer function we can get the primordial power spectrum of scalar perturbations $\mathscr{P}_R(k)$ from now-a-days observation

Transfer function

•Discussion on Perturbations from R.D to M.D

Referring to David Tong's lecture notes [**Tong**], here we discuss the evolution of matter perturbations using matter perturbations as an example.

Formally, we can write it as $\delta(\mathbf{k},t_0) = T(k)\,\delta(\mathbf{k},t_i)$. Based on the discussions in Part 3, we know that matter perturbations have different evolution inside and outside the horizon. In the R.D. era, outside the horizon, $\delta \sim a^2$, while in the M.D. era, outside the horizon, $\delta \sim a$. Therefore, a perturbation in the R.D. era corresponds to an observation in the M.D. era, which can be expressed in terms of a as:

$$\delta(\mathbf{k},t_0) = \left(\frac{a_{\text{eq}}}{a_i}\right)^2 \frac{a_0}{a_{\text{eq}}}\,\delta(\mathbf{k},t_i)$$

Furthermore, considering a shorter wavelength that enters the horizon during the R.D. era, we obtain $\delta \sim \log a$, which can be approximated as a constant.

Therefore, it can be written as

$$\delta(\mathbf{k},t_0) = \left(\frac{a_{\text{enter}}}{a_{\text{eq}}}\right)^2 \times \left[\left(\frac{a_{\text{eq}}}{a_i}\right)^2 \frac{a_0}{a_{\text{eq}}}\right]\delta(\mathbf{k},t_i)$$

Considering the R.D $a \sim t^{\frac{1}{2}}$, since $k = \frac{2\pi}{c}(aH)_{\text{enter}}$, we have $k \sim (aH)_{\text{enter}} \sim 1/a_{\text{enter}}$, so $T(k) \sim const \times k^{-2}$.

Note that different fluid components will have different transfer functions.

•Transfer function in the Sachs-Wolfe regime

Here we consider large-scale perturbations. During recombination, these perturbations are outside the horizon and do not affect the CMB. Therefore, their evolution is not affected by the complex dynamics within the sub-horizon scales. Their impact on the CMB is only through geometric projection. The mathematical calculation tells us that the transfer function is given by

$$\Delta_{T\ell}(k) = \frac{1}{3} j_\ell(k[\tau_0 - \tau_{\text{rec}}])$$

The computation yields

$$\ell(\ell+1)C_\ell^{TT} \propto \left.\Delta_{\text{s}}^2(k)\right|_{k\approx\ell/(\tau_0-\tau_{\text{rec}})} \propto \ell^{n_{\text{s}}-1}$$

Third-order correlation functions

Upon closer inspection, we can see that we are actually computing second-order correlation functions.

$$\langle a_{\ell m}^* a_{\ell' m'}\rangle = C_\ell^{TT}\delta_{\ell\ell'}\delta_{mm'} \quad \Leftrightarrow \quad \langle \mathscr{R}_{\mathbf{k}}\mathscr{R}_{\mathbf{k}'}\rangle = (2\pi)^3 P_{\mathscr{R}}(k)\,\delta(\mathbf{k}+\mathbf{k}')$$

If the perturbations are Gaussian, then the information can be fully contained in the second-order correlation function. But are the fluctuations truly Gaussian? If this is the case, we cannot only rely on two-point correlation functions for calculations. Instead, we choose higher-order correlation function such as using the **Bispectrum**.

$$(R_{\mathbf{k}_1}R_{\mathbf{k}_2}R_{\mathbf{k}_3}) = (2\pi)^3 B_{\mathscr{R}}(k_1,k_2,k_3)\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$$

since the three momenta satisfy momentum conservation and correspond to using a triangle in momentum space to scan through all the sky. Let's introduce a model that describes local non-Gaussianity.

$$\mathscr{R}(\mathbf{x}) = \mathscr{R}_g(\mathbf{x}) + \frac{3}{5} f_{NL}^{R} \star \mathscr{R}_g^2(\mathbf{x})$$

where $f^{\mathscr{R}_{NL}}$ represents the level of non-Gaussianity. The corresponding bispectrum is written as

$$B_R(k_1,k_2,k_3) = \frac{6}{5} f_{\mathrm{NL}}^{R} \left[P_R(k_1)P_R(k_2) + P_R(k_2)P_R(k_3) + P_R(k_3)P_R(k_1) \right]$$

This result of the non-Gaussianity corresponds to the implicit of the violation of the slow-roll-single-field inflation. Current observations indicate $4 f_{\mathrm{NL}} 80$ at 95% confidence level. Which means the experimental threshold for non-Gaussianity is currently set very low.

4.2 Polarization

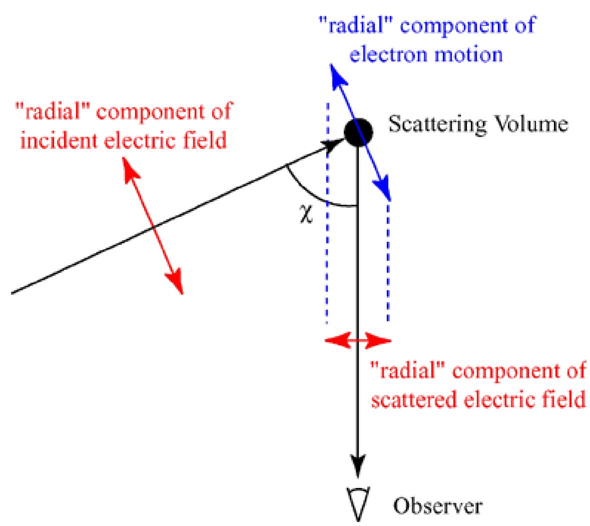


Figure 4.2: Thomson scattering

Polarization of photons is a characteristic of observed photon in CMB. How are photons polarized? Well the mechanism are shown in the picture at front. This image reflects that if the early universe is not isotropic or homogeneous and there would be photons (required to have quadrupole term) have in-homogeneous intensity. When photons collide with electrons from different directions with different strength, they would emit a polarized light. Therefore, observing the degree of polarization of photons can serve as an indication of the in-homogeneity in the early universe.

4.2.1 Quantitative Description of Polarization

I'll just copy baumann's lecture note to show some mathematical definition of the measuring tools when we observe polarization. By choosing a basis of two perpendicular unit vectors $\hat{e}_1$ and $\hat{e}_2$ orthogonal to $\hat{n}$, we can describe the intensity using a 2×2 **Intensity tensor** $I_{ij}(\hat{n})$. We define $Q \equiv \frac{1}{4}(I_{11} - I_{22})$ and $U \equiv \frac{1}{2} I_{12}$.

•Expansion in terms of spherical harmonics for the intensity combination

Claim: $Q+iU$ is a spin-2 field, $(Q+iU)(\hat{n}) \to e^{\mp 2i\psi}(Q+iU)(\hat{n})$. Therefore, we can expand it in terms of spherical harmonics as $(Q\pm iU)(\hat{n}) = \sum_{\ell.m} a_{\pm 2,\ell m \pm 2} Y_{\ell m}(\hat{n})$.

4.2.2 E and B Mode

•Reorganizing coefficients to obtain spin-0 quantities

$$a_{E,\ell m} \equiv -\frac{1}{2}(a_{2,\ell m} + a_{-2,\ell m})$$

$$a_{B,\ell m} \equiv -\frac{1}{2i}(a_{2,\ell m} - a_{-2,\ell m})$$

Definition 4.2.1 — E and B mode.

$$E(\hat{n}) = \sum_{\ell.m} a_{E,\ell m} Y_{\ell m}(\hat{n})$$

$$B(\hat{n}) = \sum_{\ell.m} a_{B,\ell m} Y_{\ell m}(\hat{n})$$

$$\text{E-mode:} \nabla \times E = 0$$

$$\text{B-mode:} \nabla \cdot B = 0$$

Since E and B mode have different behavior under parity transformation.It act differently in different perturbation.

•Scalar perturbations only have E-mode.

•Vector perturbations only have B-mode (quickly decay).

•Tensor perturbations have both B-mode and E-mode (thus B-mode is an important observational indicator of tensor perturbations).

4.2.3 Cross-correlation Observable

If we observe the second-order correlation function, we have six observable quantities, but due to the anti-symmetry of TB and EB, only TT, EE, BB, and TE can be observed:

$$C_\ell^{XY} \equiv \frac{1}{2\ell+1} \sum_m \langle a^*_{X,\ell m} a_{Y,tm} \rangle$$

TE mode:

We can observe $\ell = 50 - 200$, where $(\ell+1)C_\ell^{TE} < 0$. This claim implies adiabatic fluctuation.

4.3 Large-Scale Structure (LSS) Observables

We can observe the galaxy power spectrum, or more precisely, the dark matter (DM) power spectrum. To establish the connection between this quantity and the primordial curvature perturbation, we need the Dark Matter Transfer Function. Without considering the perturbation from horizon exit and re-entry, we can obtain:

$$T_{\delta}(k) \approx \begin{cases} 1 & k k_{\mathrm{eq}} \\ \left(k_{\mathrm{eq}}/k\right)^2 & k k_{\mathrm{eq}} \end{cases}$$

4.4 Coherent Phase

e, It is an observational evidence for Inflation in the early Universe! If the perturbation was completely random, The data shall be white noise right? How ever we can see from the C_{ℓ}^{TT} plot that there are resonance peak which means there were some non-trivial physics occurred.

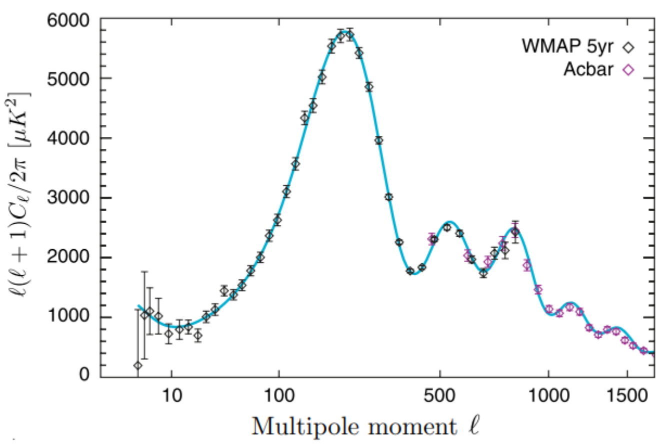


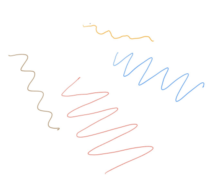
Figure 4.3: ACOUSTIC OSCILLATIONS

It is a common fact that a periodic perturbation can be expanded into sine and cosine modes. When considering the re-entry of fluctuations, since the perturbations are almost stationary on super-horizon scales $\dot{\delta} \approx 0$ the modes obtained at re-entry are all cosine modes.

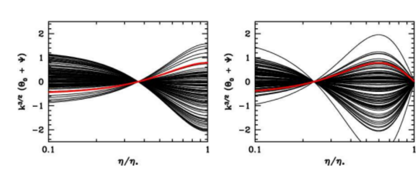
Different modes will re-enter the horizon and undergo **acoustic oscillations** within the horizon, which is equivalent to taking a snapshot at recombination. Therefore, different k-modes will have different phases at recombination.

The real picture is that different modes are excited in the universe. Modes can be classified by their k, and modes with the same k will have the same phase since they re-enter the horizon simultaneously. Therefore, these modes are coherent.

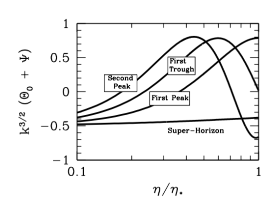
Above explanation tells us why there are many peaks in CMB spectrum. On a second thought, observing those peaks in the TT-spectrum can be considered as a good evidence for inflation.



(a) same k mode



(b) same k with same phase



(c) different k

Figure 4.4: Different k-mode

4.4.1 Lyth Bound

As mentioned in last chapter, we have two different perturbation mode. Tensor-mode and Scaler mode. Using their ratio we can define Lyth Bound as a parameter to define which kind of inflation occurred.

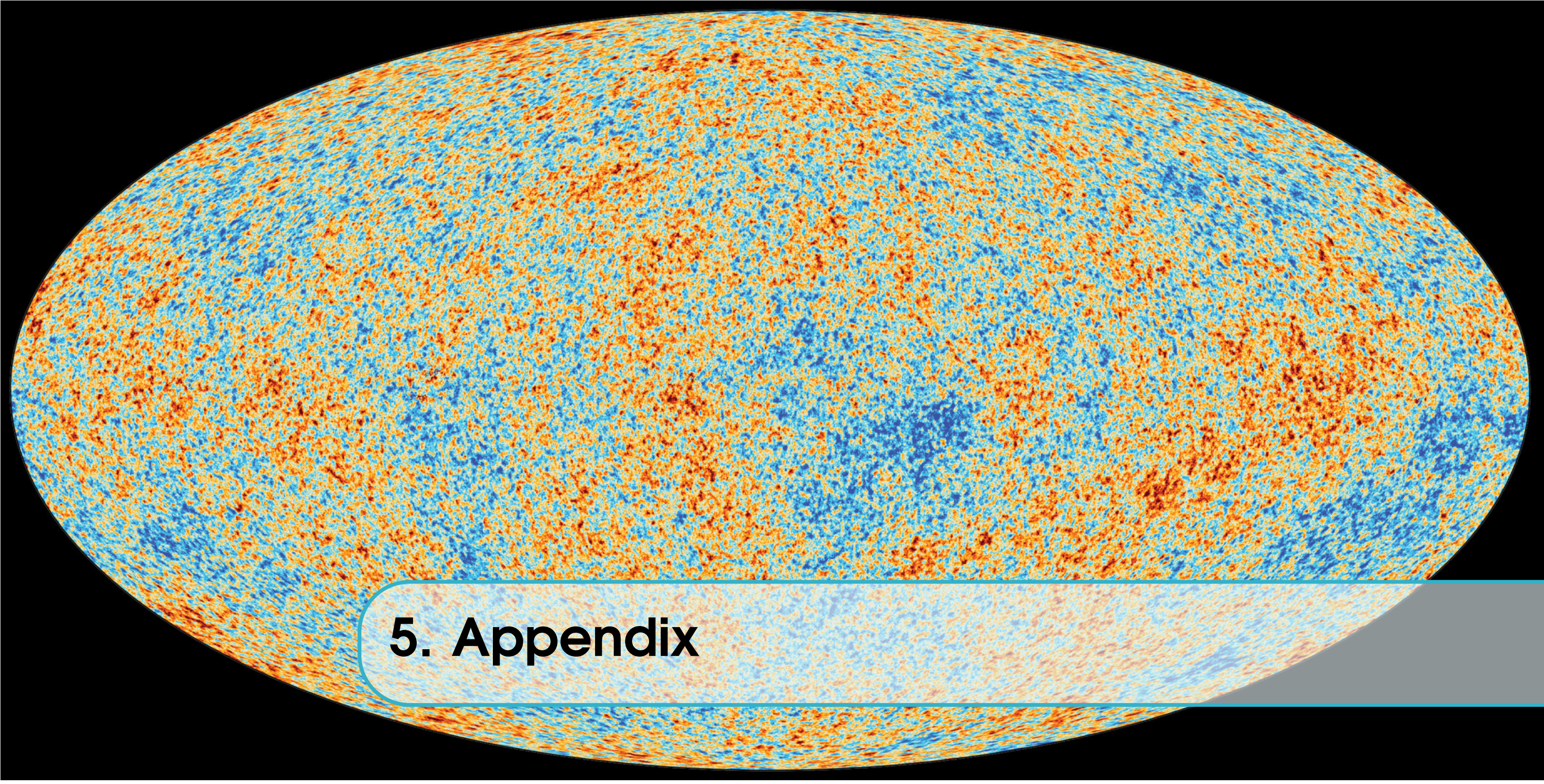
$$\Delta_t^2(k) = \frac{2}{\pi^2}\frac{H^2}{M_{\mathrm{pl}}^2}\bigg|_{k=aH}$$

$$\Delta_{\mathscr{R}}^2(k) = \frac{1}{8\pi^2}\frac{1}{\varepsilon}\frac{H^2}{M_{\mathrm{pl}}^2}\bigg|_{k=aH}$$

Since we have tensor perturbations and scalar perturbations, the latter of which carries the Hubble slow-roll parameter, the ratio of these two observable quantities can provide direct information about slow-roll:

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_{\mathscr{R}}^2(k)} = 16\varepsilon = \frac{8}{M_{pl}^2}(\frac{d\phi}{dN})^2$$

Note: Since $\frac{\dot{\phi}}{H} = \frac{d\phi}{dt\cdot H} = \frac{d\phi}{dN}$Therefore, we can obtain $\frac{\Delta\phi}{M_{\mathrm{pl}}} = \int_{N_{\mathrm{emd}}}^{N_{\mathrm{cmb}}} \mathrm{d}N\sqrt{\frac{r}{8}}$. Since $r(N)$ changes very little during the evolution, it can be treated as a constant. By observing r, we can determine the type of inflation field.



5. Appendix

5.1 Energy-Momentum Tensor

Definition 5.1.1 — Energy-Momentum Tensor.

$$T^{\mu}{}_{\nu} = (\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - \bar{P}\delta^{\mu}_{\nu}$$

Detailed derivation of $T^{\mu}{}_{\nu}$ can be found in Baumann's lecture on page 19[Bau22]. And the general idea will be introduced in the following paragraph.

Since $T_{\mu\nu}$ has homogeneity and isotropy, where isotropy implies that a comoving observer will see $T_{0i} = T_{j0} = 0$. More than that, Isotropy also implies that at $x = 0$, $T_{ij} \propto \delta_{ij} \propto g_{ij}(x = 0)$.The discussion about $x = 0$ arises because the FLRW metric rewrites the form of the metric, introducing the position x in the metric.

Homogeneity, being spatially isotropic, ensures that the relationship between $T_{\mu\nu}$ and $g_{\mu\nu}$ differs only by a time-dependent coefficient, which holds at any point in spacetime.For a general observer that is not comoving, we have $T^{\mu}{}_{\nu} = (\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - \bar{P}\delta^{\mu}_{\nu}$.

In short, when deriving $T^{\mu}{}_{\nu}$, isotropy is assumed, which eliminates all off-diagonal terms. Considering that observers can be in motion, the expanded form of $T^{\mu}{}_{\nu}$ introduces U^{μ} to describe boosts.

Definition 5.1.2 — Perturbations in the Energy-Momentum Tensor. When perturbing the energy-momentum tensor, isotropy is no longer assumed, and non-isotropic terms Π^{μ}_{ν} appear.

$$\delta T^{\mu}{}_{\nu} = (\delta\rho + \delta P)\bar{U}^{\mu}\bar{U}_{\nu} + (\bar{\rho} + \bar{P})(\delta U^{\mu}\bar{U}_{\nu} + \bar{U}^{\mu}\delta U_{\nu}) - \delta P\delta^{\mu}_{\nu} - \Pi^{\mu}_{\nu}$$

Since $\delta^{\mu}_{\nu} - \Pi^{\mu}_{\nu}$ is present, we can always redefine P to make Π^{μ}_{ν} traceless. Claim: Π^{μ}_{ν} can be chosen to be orthogonal to U^{μ}; Π^{0}_{0}, Π^{0}_{i} can be set to zero. Therefore, the anisotropic term can be neglected.

Derivation of $T_{\mu\nu}$

In the variation, there is a perturbation term δU^{μ}, so we need to find δU^{μ} first and then substitute it into the equation to solve.δU^{μ} is determined by the perturbation of the metric:

$$\begin{cases} g_{\mu\nu}U^{\mu}U^{\nu} = 1 \\ \bar{g}_{\mu\nu}\bar{U}^{\mu}\bar{U}^{\nu} = 1 \end{cases}$$

Without loss of generality, assume that $\bar{U}$ is in comoving coordinates, so $\bar{U}^{\mu} = a^{-1}\delta^{0}_{\mu}$
Since $\delta g_{00} = 2a^2$,Therefore, $\delta U^0 = -Aa^{-1}$.If we define $U^i = v^i/a$, we can have velocity:

$$\begin{cases} U^{\mu} = a^{-1}[1-A, v^2]. \\ U_{\mu} = a[1+A, -(v_i+B_i)]. \end{cases}$$

5.2 Thoughts about quantization and the relation to classic

What is canonical quantization, why do we use creation and annihilation operators to represent variables, and is this representation unique?

5.2.1 Canonical Quantization and Creation and Annihilation Operators

To understand canonical quantization, it is helpful to refer directly to the model of a harmonic oscillator. After all, all quantization processes are "homomorphic" to the quantization process of a harmonic oscillator. The quantization of a harmonic oscillator can be seen as a special case, but the essence is that the uncertainty relation leads to the emergence of quantum properties in fields.

•So, how is a harmonic oscillator quantized?

Referring to Griffiths' textbook, the so-called quantization process is simply solving the Schrödinger equation for a harmonic oscillator. We can solve it directly, using Hermite polynomials to calculate explicitly. The final result shows that the eigenfunctions of the Hamiltonian, corresponding to the eigenvalues E_n, are discrete, which is the so-called quantization.

We find a mathematical trick called $[a, a^{\dagger}] = 1$, which provides a more convenient way to describe the process of finding eigen functions. We can use combinations of creation and annihilation operators to represent all states with positive energy, as

$$\hat{H} = w(\hat{a}^{\dagger}\hat{a} + \frac{1}{2})$$

.

Mathematically, we find the commutation relation for creation and annihilation operators, which naturally allows us to classify quantum states. This process is called quantization. Therefore, the core of quantization lies in the existence of commutation relations.

•This commutation relation is not arbitrary; it must come from somewhere.

It stems from:

$$[\hat{x}, \hat{p}] = i$$

If we say that this comes directly from the definition of momentum in macro and micro scales, I think it is too hasty (for specific steps, refer to Griffiths' textbook, it is indeed easy to understand and calculate, but it always gives a feeling of being forced and patched).

I prefer starting from the uncertainty principle: the derivation of the generalized uncertainty relation relies only on the basic assumptions of quantum mechanics - that there is a Hilbert space and an inner product space, which guarantees the Cauchy inequality. Thus, we can ensure that

$$\sigma_A^2 \cdot \sigma_B^2 \geq |\psi|(A^* - \langle A^* \rangle) \cdot (B - \langle B \rangle)\psi|^2$$

where σ_X is the variance of X

Mathematical operations tell us that

$$|\langle f \cdot g \rangle|^2 \geq |\frac{1}{2i}\langle [A,B] \rangle|^2$$

.

So, perhaps experiments tell me that

$$[\hat{x}, \hat{p}] \neq 0$$

This uncertainty relation is general, so the numerical solution can be obtained by calculating i in the one-dimensional case.

Is the quantization method unique?

The quantization process in Baumann's note is different from directly constructing $\hat{a}\,\hat{a}^\dagger$.

Similarly, we only need to consider the harmonic oscillator,

$$S[q] = \frac{1}{2}\int \mathrm{d}t \left[\dot{q}^2 - \omega^2 q^2\right]$$

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

Note:

It seems to derive the commutation relation from the action of the harmonic oscillator, but it does not specifically depend on the form of the potential. It only requires

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

Impose a canonical condition $[\hat{q}, \hat{p}] = i$

$$\hat{q}(t) = q(t)\,\hat{a} + q(t)^*\,\hat{a}^\dagger$$

$$\hat{p} \equiv \dot{\hat{q}}(t) = \dot{q}(t)\,\hat{a} + \dot{q}(t)^*\,\hat{a}^\dagger$$

So we must have

$$W[q, q^*] \times [\hat{a}, \hat{a}^\dagger] = 1$$

Here, the Wronskian is defined as

$$W[q_1, q_2^*] \equiv -i\,(q_1 \partial_t q_2^* - (\partial_t q_1)\,q_2^*)$$

and it is stipulated that

$$W[q_1, q_2^*] = 1$$

It naturally follows that

$$[\hat{a}, \hat{a}^\dagger] = 1$$

Compared to the definition of the harmonic oscillator, we have more freedom here, and we can see that the canonical momentum is not completely determined.

We need to add some conditions to determine it. For example, we can choose the vacuum state as the ground state. One of the prerequisites for the vacuum state to be the ground state is that applying $\hat{H}$ should still yield 0.

So we have

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$$

$$\hat{H}|0\rangle = \frac{1}{2}(\dot{q}^2 + \omega^2 q^2)^* \hat{a}^\dagger \hat{a}^\dagger |0\rangle + \frac{1}{2}(|\dot{q}|^2 + \omega^2 |q|^2)|0\rangle$$

So we require

$$\dot{q}^2 + \omega^2 q^2 = 0$$

and we can solve the operator equation in the Heisenberg picture.

Some may say that this is the harmonic oscillator equation in the Heisenberg picture, right?

Note:

This is inconsistent with the quantization and solution process in Griffiths' book and other classical quantum mechanics textbooks. The solution process in Griffiths' book relies on quantization in the Schrödinger picture first, and then performs a change of representation. Here, quantization is directly performed in the Heisenberg picture.

Why does the lack of canonical variable commutation relations lead to a classical system?

What is the correspondence between this relation and commutation relations in classical mechanics?

We know that the derivation of canonical variables depends on the action and is independent of quantum-related processes. In classical mechanics, the evolution of canonical variables still satisfies the Poisson equation, and any macroscopic physical quantity can be described using canonical coordinates.

In classical mechanics, physical quantities evolve with time according to the equation:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}_{PB}$$

In quantum mechanics, the Heisenberg equation describes the system's time evolution based on commutation relations:

$$\frac{d\langle F\rangle}{dt} = \langle\frac{\partial F}{\partial t}\rangle + \langle[F, H]\rangle$$

5.3 From metric to SVT

Some basic fact about SVT's component: It is a useful describing tools and the mgnitude changes with speed.

•SVT has the ability to describe certain things:

∗Scaler part of SVT fits to describe Newtonian gravity.

∗Vector part of SVT fits to describe gravitomagnetism, which is a gravitational field generated by a rapidly rotating source induces a rotational effect, producing a magnetic field.

∗Tensor fits to describe gravitational radiation

•Magnitude properties:

If the source moves with $\frac{v}{c}$, then the vector and tensor correspond to $O(v/c)$ $O(v/c)^2$.

5.3.1 Metric

Since we are interested in three-dimensional space, our metric focuses on three dimensions. Here, we choose a metric corresponding to the discussion of three-dimensional space under constant conformal time.

We choose the tensor form so that it remains invariant under coordinate transformations and also remains invariant when switching coordinate systems, such as switching from spherical coordinates to hyperbolic coordinates. Thus, we adopt a coordinate-independent representation.

Vector:

$$\mathbf{A} = A^i e_i$$

Tensor:

$$\mathbf{h} = h_{ij} e^i \otimes e^j$$

Basis vectors:

$$e_i \cdot e_j = \gamma_{ij}$$

$\nabla operator:$

Covariant derivative:

$$\nabla_i \gamma_{ijk} = 0$$

The changes brought about by the coordinate-independent form are when $K \neq 0$:

∗1.The ∇ operator looks much more complex.

$$\gamma \equiv \det\{\gamma_{ij}\} \varepsilon^{ijk} = \gamma^{-1/2} [ijk][ijk] = \pm 1$$

$$\begin{cases} \nabla^2 \phi \equiv \gamma^{-1/2} \partial_i \left(\gamma^{1/2} \gamma^{ij} \partial_j \phi \right) \\ \nabla \cdot v \equiv \gamma^{-1/2} \partial_i \gamma^{1/2} v^i \\ \nabla \times w \equiv \varepsilon^{ijk} (\partial_i v_j) e_k \end{cases}$$

$*$2.The ∇ operator becomes non-commutative. The introduction of coordinates relative to the observer in the FLRW metric is a major reason for the non-commutativity of ∇.

$$\cdot [\nabla_j, \nabla_k] \equiv (\nabla_j \nabla_k - \nabla_k \nabla_j)$$

Corresponding to the vector and tensor, we have the following relations:

$$\begin{cases} [\nabla_j, \nabla_k] A^i = {}^{(3)}R^i{}_{njk} A^n \\ [\nabla_j, \nabla_k] h^{ij} = {}^{(3)}R^i{}_{nkl} h^{nj} + {}^{(3)}R^j{}_{nkl} h^{in} \end{cases}$$

Riemann curvature tensor in three-dimensional space:

$$^{(3)}R_{\ jkl} = K\left(\delta^i{}_k \gamma_{jl} - \delta^i{}_l \gamma_{jk}\right)$$

5.3.2 SVT

•Perturbed form of the FLRW metric

$$\begin{cases} ds^2 = a^2(\tau)\left\{-(1+2\psi)d\tau^2 + 2w_i d\tau dx^i + [(1-2\phi)\gamma_{ij} + 2h_{ij}]\, dx^i dx^j\right\} \\ \gamma^{ij} h_{ij} = 0 \end{cases}$$

From the components inside, we can see that:
$*$There are two scalar fields ψ and ϕ;
$*$One vector field $w_i e^i$;
$*$One traceless tensor field $h_{ij} e^i \otimes e^j$
Corresponding to $DOF = 10$
$\sum_{m=-\ell^{m=\ell}}$
The focus of SVT research is on the spatial part. The theory served by SVT is mainly linear perturbation theory, which means that only first-order terms need to be consideredSure! I can provide you with an overview of the Scalar-Vector-Tensor (SVT) decomposition in the context of gravitational perturbations.

SVT can be considered as a generalization of vector decomposition

•Review Vector Decomposition:

$w = w_\parallel + w_\perp$ is a decomposition of vector w,$\nabla \times w_\parallel = \nabla \cdot w_\perp = 0$. Recall from vector analysis that the curl of a gradient is zero, and the divergence of a curl is zero. Therefore, $w_\perp$ corresponds to the vector perturbation, and $w_\parallel$ corresponds to the scalar perturbation. This distinction is based on their fundamental degrees of freedom. •Tensor Decomposition:

We only consider tensors that can be decomposed into two vectors using "$\otimes$". Thus, we can understand $h_{ij} = w_\perp \otimes w_\perp + w_\perp \otimes w_\parallel + w_\parallel \otimes w_\parallel$. Here, "$\otimes$" represents the independence of the two quantities (this explanation is very physical). Therefore, when acting on "∇," we can separate each position.Thus We can find h, $\mathbf{h}$, and $h^i{}_j$ such that any tensor can be decomposed into these three combinations with ∂:

$$\begin{cases} h(x) = h_\perp + h_\parallel + h_T \\ h_{ij,\parallel} = D_{ij} h \\ h_{ij,\perp} = \nabla_{(i} h_{j)} \\ \nabla_i h^i{}_{j,\top} = 0 \end{cases}$$

Here, $\nabla_{(i}h_{j)} \equiv \frac{1}{2}\left(\nabla_i h_j + \nabla_j h_i\right); D_{ij} \equiv \nabla_i \nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2$

The reason for this classification is that these three types of tensor elements have different responses under the action of ∇. The inclusion of K in the equation is likely due to the relationship $[\nabla_i, \nabla_j] = f(K)$, but the derivation is somewhat difficult and won't be demonstrate here.

Therefore, we have found two modes of vibration:

$*$longitudinal mode:

$$\nabla \cdot h_{\parallel} = \frac{2}{3}\nabla\left(\nabla^2 + 3K\right)h$$

$*$solenoidal:

$$\nabla \cdot h_{\perp} = \frac{1}{2}\left(\nabla^2 + 2K\right)\mathbf{h}$$

Discussion on the uniqueness of the SVT decomposition:

Unfortunately this decomposition is not unique, except for adding a constant factor.

For Vector part:

h can also differ by a Killing Vector field, which satisfies

$$\nabla_i h_j + \nabla_j h_i = 0$$

Without loss of generality, considering this equation in Cartesian coordinates, a possible solution is $(h_x, h_y, h_z) = (y, -x, 0)$. If in a universe where $K \leq 0$, which corresponds to an unbound universe, the corresponding Killing Vector Field will diverge with distance (seems pretty similar to renormalization, hahah).

If $K \geq 0$, corresponding to a closed universe, this quantity is finite. In this case, the Killing Vector Field is equivalent to spatial rotation and is not a physical perturbation.

So the Killing Vector Field corresponds to a non-physical quantity.

Tensor part:

Our goal is to find the corresponding kernel in the equation of the form

$$h_{ij,\mathrm{T}} \to h_{ij,\mathrm{T}} + \zeta_{ij}\ , \quad \zeta_{ij} \equiv \left[\nabla_i \nabla_j - \gamma_{ij}\left(\nabla^2 + 2K\right)\right]\zeta \Rightarrow \zeta_{ij} = D_{ij}\zeta$$

It can be proven that $\nabla_i \zeta^i_j = 0$, so adding ζ in the tensor perturbation part is also a solution.

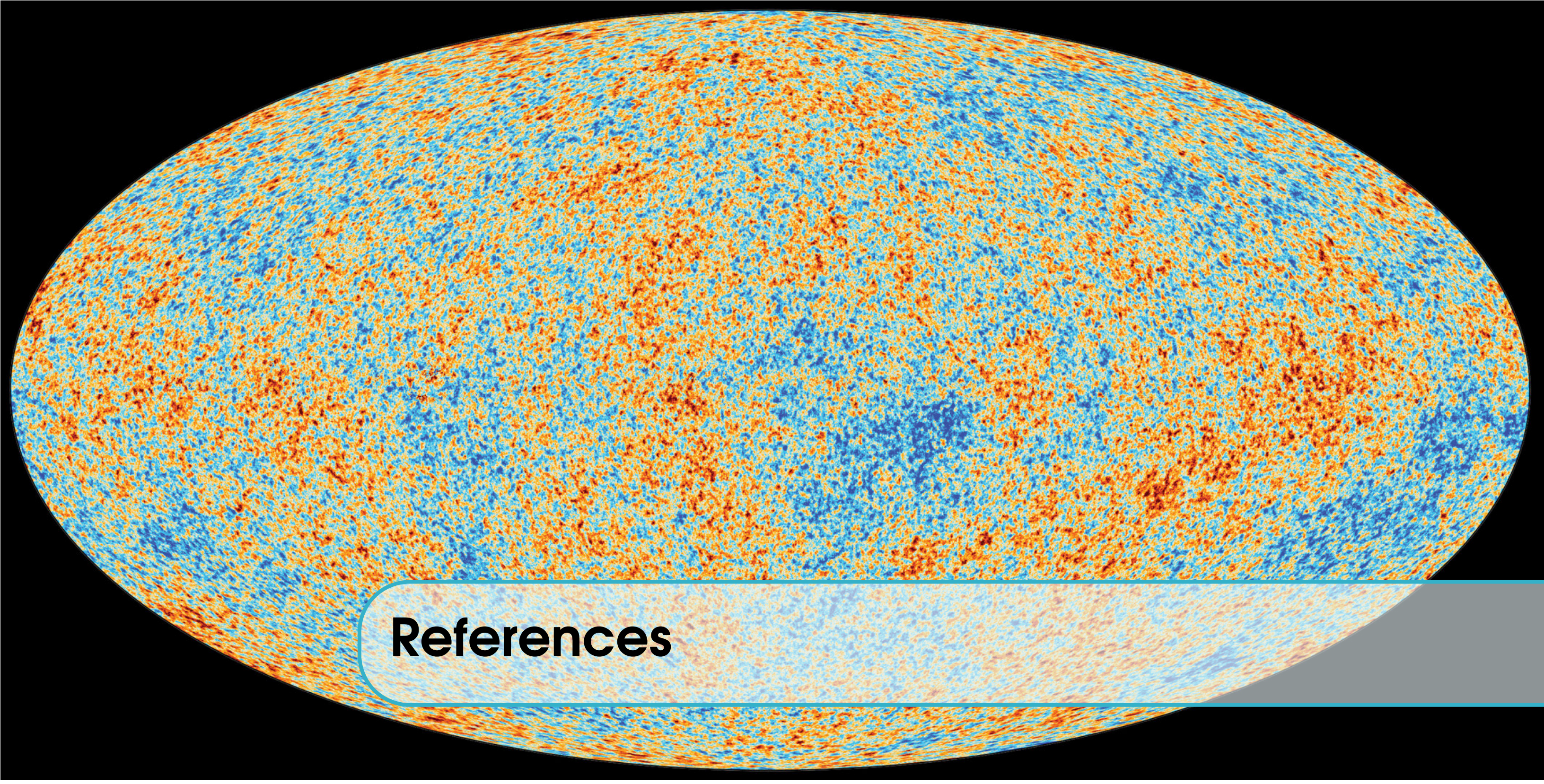
The kernel part perturbation of h_{ij} can be merged into $h_{\parallel}$.It can be observed that this kernel is actually a part of the scalar perturbation, so it can always be merged into the scalar perturbation. Therefore, we define that there is no $D_{ij}\zeta$ term in the tensor perturbation.

Since h_{ij} satisfies the traceless condition. Since we can merge the trace perturbation into ϕ, we can obtain the restriction condition for the kernel part perturbation.

$$\gamma^{ij}(\left[\nabla_i \nabla_j - \gamma_{ij}\left(\nabla^2 + 2K\right)\right]\zeta) = 0 \Rightarrow (\nabla^2 + 3K)\zeta = 0$$

•If $K \leq 0$, considering the case where it does not diverge with space, there is only the trivial solution.
•If $K \geq 0$, there are four independent solutions. Degrees of freedom (DOF) of tensor perturbations: Initial: DOF=10 Symmetry: DOF-4 Traceless: DOF-1 $\nabla_i h^i_{j,T} = 0$: DOF-3 Finally, DOF=2

The physical meaning of these solutions: Correspond to the redefinition of spacetime coordinates, which has no physical significance.



References

bibliography